

r -Indecomposable and r -Nearly Decomposable Matrices *

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AMS Subject Classification: 15A36, 05C20, 05C50

Abstract

Let n, r be integers with $0 \leq r \leq n - 1$. An $n \times n$ matrix A is called r -partly decomposable if it contains a $k \times l$ zero submatrix with $k + l = n - r + 1$. A matrix which is not r -partly decomposable is called r -indecomposable (shortly, r -inde). Let E_{ij} be the $n \times n$ matrix with a 1 in the (i, j) position and 0's elsewhere. If A is r -indecomposable and, for each $a_{ij} \neq 0$, the matrix

*Research supported by National Natural Science Foundation of China (No.10331020) and NNSF of Guangdong Province (No.04010389).

$A - a_{ij}E_{ij}$ is no longer r -indecomposable, then A is called r -nearly decomposable (shortly, r -nde). In this paper, we derive numerical and enumerative results concerning r -nde matrices of 0's and 1's. We also obtain some bounds on the index of convergence of r -inde matrices, especially for the adjacency matrices of primitive Cayley digraphs and circulant matrices. Finally, we propose an open problem for further research.

Keywords: partly decomposable, indecomposable, nearly decomposable, primitive matrix, exponent, Cayley digraph

1 Introduction

Let B_n denote the set of all matrices of order n over the Boolean algebra $\{0, 1\}$. Let $J_n \in B_n$ be the matrix whose entries are all equal to 1, and let $E_{ij} \in B_n$ be the matrix with a 1 in the (i, j) position and 0's elsewhere. There is a one to one correspondence between B_n and all digraphs $D = (V, E)$ with vertex set $V = \{1, 2, \dots, n\}$ and arc set $E = \{(i, j) \in V \times V : a_{ij} = 1\}$. The digraph D corresponding to $A \in B_n$ is called the *associated digraph* of A , denoted by $D(A)$; and the matrix A corresponding to D is called the *adjacency matrix* of D , denoted by $A(D)$. A digraph D is *strong* if, for any two vertices x and y , D contains a path from x to y and a path from y to x . A matrix $A \in B_n$ is *irreducible* if its associated digraph is strong.

Let n, r be integers with $0 \leq r \leq n - 1$. An $n \times n$ matrix A is called *r -partly decomposable* if it contains a $k \times l$ zero submatrix with $k + l = n - r + 1$. A matrix which is not r -partly decomposable is called *r -indecomposable* (shortly, r -inde). If a matrix A is r -inde and, for each $a_{ij} \neq 0$, the matrix $A - a_{ij}E_{ij}$ is no longer r -indecomposable, then A is called *r -nearly decomposable* (shortly, r -nde). In particular, a 0-inde matrix is called a Hall matrix, and a 1-inde (resp. 1-nde) matrix is called a fully indecomposable (resp. nearly decomposable) matrix. If A is r -inde (resp. r -nde), its associated digraph $D(A)$ is called r -inde (resp. r -nde).

A matrix A is said to have a positive diagonal if there exists a permutation φ of $\{1, 2, \dots, n\}$ such that all entries $a_{1\varphi(1)}, a_{2\varphi(2)}, \dots, a_{n\varphi(n)}$ are positive. If $a_{ii} > 0$ for all i , then A is said to have a positive main diagonal, denoted by $I \leq A$.

Let $A = (a_{ij}) \in B_n$, and let s and t be integers with $1 \leq s, t \leq n$. For $1 \leq i_1 < i_2 < \dots < i_s \leq n$ and $1 \leq j_1 < j_2 < \dots < j_t \leq n$, we write $\alpha = (i_1, i_2, \dots, i_s)$ and $\beta = (j_1, j_2, \dots, j_t)$. Let $A[i_1, i_2, \dots, i_s | j_1, j_2, \dots, j_t] =$

$A[\alpha|\beta]$ denote the submatrix of A whose (p, q) entry is a_{ipjq} ; that is, $A[\alpha|\beta]$ is obtained from A by deleting those rows not indexed in α and columns not indexed in β . Similarly, $A(\alpha|\beta)$ denotes the matrix obtained from A by deleting the rows not indexed in α and columns indexed in β . The matrices $A(\alpha|\beta)$ and $A[\alpha|\beta]$ are defined analogously. Let $\lfloor x \rfloor$ denote the maximum integer s such that $s \leq x$, and let $\lceil x \rceil$ denote the least integer s such that $s \geq x$.

Let $n \geq m \geq 1$ be integers and let $A = (a_{ij}) \in B_{m \times n}$. The permanent of A is

$$Per(A) = \sum_{(i_1, i_2, \dots, i_m) \in P_m^n} a_{1i_1} a_{2i_2} \dots a_{mi_m},$$

where P_m^n is the set of all m -permutations of $\{1, 2, \dots, n\}$.

For an $n \times n$ Boolean matrix A , the behavior of the sequence A, A^2, A^3, \dots mainly depends on two parameters: the period of A and the index of convergence of A . The period of A , denoted by $p(A)$, is the least positive integer p such that $A^{l+p} = A^l$ for some l , and the index of convergence of A , denoted by $k(A)$, is the least value of l for which $A^{l+p} = A^l$ holds. If A is irreducible with $A \neq [0]$ and $p(A) = 1$, we call A primitive, and in which case the index of convergence of A is called the exponent of A , denoted by $\exp(A)$. It is well known that A is primitive if and only if there exists a positive integer k such that $A^k = J$.

Let $\sigma(A)$ be the number of entries of A equal to 1, and let $m(D)$ be the number of arcs in $D(A)$. Then $\sigma(A) = m(D)$. Let

$$f(n, r) = \max\{\sigma(A) : A \in B_n \text{ is } r\text{-nde}\}$$

and

$$g(n, r) = \min\{\sigma(A) : A \in B_n \text{ is } r\text{-nde}\}.$$

Clearly, a matrix $A \in B_n$ is $(n-1)$ -inde or $(n-1)$ -nde if and only if $A = J$. So throughout the paper we always assume $r < n-1$ and only discuss this non-trivial case. In [6], Brualdi and Hedrick derived some structural, numerical, and enumerative results concerning nearly decomposable matrix of 0's and 1's. For $r = 1$ (and $n > r + 1 = 2$), they proved that

$$f(n, 1) = 3n - 3 \text{ and } g(n, 1) = 2n.$$

In this paper, we continue the study of the two functions $f(n, r)$ and $g(n, r)$ for $r \geq 2$. The paper is organized as follows. In Section 2, we give some sufficient and necessary conditions on r -inde matrices by means of permanent

and r -connected digraph (r -irreducible matrix), and give some examples for r -inde matrices. In Section 3, we prove that $g(n, r) = n(r + 1)$ and that

$$f(n, r) \geq f'(n, r) = \begin{cases} \lfloor \frac{(n+r+1)^2}{4} \rfloor & \text{if } n < 3r, \\ (n-r)(2r+1) & \text{if } n \geq 3r. \end{cases}$$

Moreover, for each i with $g(n, r) \leq i \leq f'(n, r)$, we construct an r -nde matrix $A \in B_n$ with $\sigma(A) = i$. This extends a result [6, Theorem 3.4] for $r = 1$ by Brualdi and Hedrick. In Section 4, we discuss the exponent of r -inde and r -nde matrices and provide a new and simpler proof for a result [11, Theorem 2.1] by Huang. In Section 5, we propose an open problem on r -nde matrices as a suggestion for further research.

2 r -Indecomposable matrices

Theorem 2.A (Frobenius-Konig) The permanent of an $n \times n$ nonnegative matrix A is zero if and only if A contains an $s \times t$ zero submatrix with $s + t = n + 1$. \square

Lemma 2.1 Suppose $0 \leq r \leq n - 1$ and $A \in B_n$. Then the following are equivalent.

- (i) The matrix A is r -inde.
- (ii) For each k with $1 \leq k \leq n - r$, the matrix A does not have any $k \times l$ zero submatrix with $k + l = n - r + 1$.
- (iii) For each k with $1 \leq k \leq n - r$, every $k \times n$ submatrix of A has at least $k + r$ nonzero columns.
- (iv) For each subset $S \subseteq V(D(A))$ with $1 \leq |S| = k \leq n - r$, the inequality $|N(S)| \geq k + r$ holds, where $N(S) = \{v \in V(D(A)) : \text{there is an arc } (u, v) \text{ for some } u \in S\}$ is the set of out-neighbors of S in $D(A)$.

Proof (i) \iff (ii) follows from the definition of r -indecomposability. The matrix interpretation of (ii) is equivalent to (iii). The graph interpretation of (iii) is equivalent to (iv). \square

It is evident that an r -inde matrix is a 1-inde matrix for any $r \geq 1$, and a 1-inde matrix is primitive. Let $B_{n,r}$ be the set of all r -inde matrices of order n .

Lemma 2.2 $\{J\} = B_{n,n-1} \subset B_{n,n-2} \subset \dots \subset B_{n,2} \subset B_{n,1} \subset B_{n,0}$, and every r -inde matrix with $r \geq 1$ is primitive. \square

Theorem 2.1 An $n \times n$ nonnegative matrix A is r -inde if and only if $Per(A(\alpha|\beta)) > 0$ for any $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ with $|\alpha| = |\beta| = r$.

Proof “ \implies ” Suppose otherwise $Per(A(\alpha|\beta)) = 0$ for some α and β . Then by Theorem 2.A, the submatrix $A(\alpha|\beta)$ and thus A contain an $s \times t$ zero submatrix with $s+t = (n-r)+1$, a contradiction to the r -indecomposability of A .

“ \impliedby ” Suppose A has an $s \times t$ zero submatrix C such that $s+t = n-r+1$. Then $s, t \leq n-r$. Thus C is a submatrix of $A(\alpha|\beta)$ for some $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ with $|\alpha| = |\beta| = r$. By Theorem 2.A, we have $Per(A(\alpha|\beta)) = 0$, a contradiction. Therefore A is r -inde. \square

Let $r \geq 1$. A digraph D with at least $r+1$ vertices is called r -connected if each digraph obtained from D by removing any $r-1$ vertices is strong. Thus an r -connected digraph has the property that, for any $r+1$ distinct vertices $i_1, i_2, \dots, i_r, i_{r+1}$, there exists a path from i_1 to i_{r+1} without containing i_2, \dots, i_r .

Theorem 2.2 Suppose $r \geq 1$ and $I_n \leq A \in B_n$. Then A is r -inde if and only if $D(A)$ is r -connected.

Proof “ \implies ” Suppose A is r -inde. By Theorem 2.1, $A(\alpha|\beta)$ is 1-inde for any $\alpha = \beta \in \{1, 2, \dots, n\}$ with $|\alpha| = |\beta| = r-1$. So $D(A)$ is r -connected.

“ \impliedby ” Suppose A is r -partly decomposable. Then there exists a $p \times q$ zero submatrix C with $p+q = n-r+1$. Since $I_n \leq A$, the zero submatrix C does not contain any entry on the main diagonal. Thus, there exist $i_1, i_2, \dots, i_p, j_1, j_2, \dots, j_{r-1} \in \{1, 2, \dots, n\}$ such that $A[i_1, i_2, \dots, i_p | i_1, i_2, \dots, i_p, j_1, j_2, \dots, j_{r-1}] = C$. Then the digraph obtained from $D(A)$ by removing the $r-1$ vertices corresponding to the j_1 -th, j_2 -th, \dots , j_{r-1} -th rows of A is not strong, a contradiction to the r -connectedness of $D(A)$. \square

A matrix $A \in B_n$ is called r -reducible if by taking a simultaneous row permutation and column permutation, A is similar to the form $\begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$,

where A_{11} and $\begin{bmatrix} A_{22} & A_{23} \end{bmatrix}$ are square matrices of order at least one and the size of O is $p \times (n-r+1-p)$. If A is not r -reducible, then A is called r -irreducible. The property of r -irreducibility has the following interpretation in terms of its associated digraph.

Lemma 2.3 Suppose $r \geq 1$. Then a matrix A is r -irreducible if and only if its associated digraph D is r -connected.

Proof “ \Leftarrow ” Suppose A is r -reducible. Then A contains a $p \times (n - r + 1 - p)$ zero submatrix. Without loss of generality, we suppose the rows of the zero submatrix correspond to vertices $1, 2, \dots, p$ and the columns correspond to vertices $p + r, \dots, n$. Let D_1 be obtained from D by deleting vertices $p + 1, \dots, p + r - 1$. Then the adjacency matrix of D_1 is reducible. Thus D_1 is not strong, a contradiction.

“ \Rightarrow ” Suppose that D is not r -connected. Then there exists a non-strong digraph D_1 obtained from D by removing $r - 1$ vertices. Let A_1 be the adjacency matrix of D_1 . Then there exists a permutation matrix P such that $PA_1P^{-1} = \begin{bmatrix} A_{11} & O \\ C & B \end{bmatrix}$, where A_{11} and B are square matrices of order at least one and O is a $p \times (n - r + 1 - p)$ zero matrix. Thus PAP^{-1} and so A contain a $p \times (n - r + 1 - p)$ zero matrix, a contradiction. \square

It is easy to see that the property of r -indecomposability is preserved under row permutations and column permutations. Suppose A is r -inde. Then there exist permutation matrices P and Q such that PAQ has a positive main diagonal and PAQ is r -inde.

Theorem 2.3 A matrix A is r -inde if and only if there exist permutation matrices P and Q such that PAQ is r -irreducible and PAQ has a positive main diagonal. \square

Now we show some examples of digraphs with high indecomposability. Let G be a multiplicative group with identity element e , and let $A = \{a_1, \dots, a_k\}$ be a subset of G . The (*right*) *Cayley digraph* is the digraph $\text{Cay}(G, A) = (V, E)$ where $V = G$ and $E = \{(x, y) : x^{-1}y \in A\}$. Thus $\text{Cay}(G, A)$ is regular of outdegree $k = |A|$.

Lemma 2.4 (Shen and Gregory [10]) Let $A = \{a_1, \dots, a_k\}$ be a subset of an Abelian group G . Then $\text{Cay}(G, A)$ is primitive if and only if $\text{Cay}(G, A_1)$ is strong, where $A_1 = \{a_i a_1^{-1} : 1 \leq i \leq k\}$. \square

Lemma 2.5 (Hamidoune [9]) Any loopless strong vertex-transitive digraph with outdegree k is $(\lfloor k/2 \rfloor + 1)$ -connected. \square

Theorem 2.4 Let $A = \{a_1, \dots, a_k\}$ be a subset of an Abelian group G . Suppose $\text{Cay}(G, A)$ is primitive. Then $\text{Cay}(G, A)$ is $\lceil k/2 \rceil$ -indecomposable.

Proof Let $A_1 = \{a_i a_1^{-1} : 1 \leq i \leq k\}$. Let M_1 and M_2 be the adjacency

matrices of $\text{Cay}(G, A)$ and $\text{Cay}(G, A_1)$, respectively. Then $M_2 = M_1P$, where P is the permutation matrix for $\sigma : g \rightarrow ga_1^{-1}$; that is, an entry of P is 1 if and only if this entry has row index g and column index ga_1^{-1} for some $g \in G$. This implies that M_2 can be obtained from M_1 by permuting columns of M_1 . Thus M_1 and M_2 have the same indecomposability, so do $\text{Cay}(G, A)$ and $\text{Cay}(G, A_1)$. Since $\text{Cay}(G, A)$ is primitive, by Lemma 2.4, we know that $\text{Cay}(G, A_1)$ is strong. Then, by Lemma 2.5, The digraph $\text{Cay}(G, A_1)$ is $\lfloor (k-1)/2 \rfloor + 1 = \lceil k/2 \rceil$ -connected. Since $\text{Cay}(G, A_1)$ has a loop at each vertex, by Theorem 2.2, it is $\lceil k/2 \rceil$ -inde. Therefore, the digraph $\text{Cay}(G, A)$ is $\lceil k/2 \rceil$ -inde. \square

Note that Theorem 2.4 may not hold for non-Abelian Cayley digraphs since Lemma 2.4 does not hold for non-Abelian Cayley digraphs. Nevertheless, the techniques in Theorem 2.4 work for non-Abelian Cayley digraphs. That is, for any group G (Abelian or non-Abelian), the Cayley digraphs $\text{Cay}(G, A)$ and $\text{Cay}(G, A_1)$ have the same indecomposability. Furthermore, if $\text{Cay}(G, A_1)$ is strong, then $\text{Cay}(G, A)$ is $\lceil k/2 \rceil$ -indecomposable.

3 Bounds on the number of 1's in r -nde matrices

Let $\sigma(A)$ be the number of entries of A equal to 1, and let $m(D)$ be the number of arcs in $D(A)$. Then $\sigma(A) = m(D)$. Let $f(n, r) = \max\{\sigma(A) : A \in B_n \text{ is } r\text{-nde}\}$ and $g(n, r) = \min\{\sigma(A) : A \in B_n \text{ is } r\text{-nde}\}$.

Example 3.1 Let N be the adjacency matrix of the Cayley digraph $\text{Cay}(Z_n, \{1, 2, \dots, r+1\})$. Then N is r -nde with $\sigma(N) = n(r+1)$.

Proof Since there exists a permutation matrix P such that $M = NP$ with $D(M) = \text{Cay}(Z_n, \{0, 1, 2, \dots, r\})$, it suffices to prove M is r -nde. By Theorem 2.2, it suffices to prove $\text{Cay}(Z_n, \{0, 1, 2, \dots, r\})$ is r -connected. We use induction to prove the latter statement. For $r = 1$, certainly $\text{Cay}(Z_n, \{0, 1\})$ is 1-connected. Now suppose $\text{Cay}(Z_{n-1}, \{0, 1, 2, \dots, r-1\})$ is $(r-1)$ -connected. We observe that the removal of any vertex from $\text{Cay}(Z_n, \{0, 1, 2, \dots, r\})$ results a superdigraph of $\text{Cay}(Z_{n-1}, \{0, 1, 2, \dots, r-1\})$ on the same vertex set. Thus the resultant superdigraph is $(r-1)$ -connected. This implies that $\text{Cay}(Z_n, \{0, 1, 2, \dots, r\})$ is r -connected. \square

The following lemma follows from the above example and the fact that an r -inde matrix has at least $r+1$ 1's in each row and each column.

Lemma 3.1 $g(n, r) = n(r+1)$. \square

Definition 3.1 Suppose $M \in B_{n-t}$ is an $(r-t)$ -nde matrix with $\sigma(M) = (r-t+1)(n-t)$. We define

$$A_t = \left[\begin{array}{c|c} O_{t \times t} & J_{t \times (n-t)} \\ \hline J_{(n-t) \times t} & M_{(n-t) \times (n-t)} \end{array} \right].$$

Clearly, $\sigma(A_t) = (n-t)(r+1+t)$.

Lemma 3.2 Suppose $n \geq r+2p$, $1 \leq p \leq r$. Then A_1, A_2, \dots, A_p are r -nde.

Proof First, we want to prove that A_t is r -inde for all t , $1 \leq t \leq p$. Let $Y = V(D(M))$, $X = V(D(A_t)) - V(D(M))$. Then $|Y| = n-t$, $|X| = t$. Let $S \subseteq V(D(A_t))$ with $1 \leq |S| = k \leq n-r$.

Case 1: $S \subseteq X$. Then $k \leq t$ and

$$|N(S)| = |Y| = n-t \geq (r+2p) - t = r+p + (p-t) \geq r+p \geq r+k.$$

Case 2: $S \subseteq Y$. Since M is $(r-t)$ -nde, by Lemma 2.1, we have $|N(S) \cap Y| \geq |S| + r - t$. Thus

$$|N(S)| \geq |S| + r - t + |X| = |S| + r - t + t = k + r.$$

Case 3: $S = S_1 \cup S_2$, where $S_1 \subseteq X$, $S_2 \subseteq Y$. Then

$$|N(S)| = |N(S_1) \cup N(S_2)| = |Y| + |X| = n \geq k + r.$$

Thus, by Lemma 2.1, A_t is r -inde. Furthermore, A_t is r -nde since, for each non-zero (i, j) entry of A_t , either row i or column j contains exactly $(r+1)$ 1's. \square

$$\text{Let } f'(n, r) = \begin{cases} \left\lfloor \frac{(n+r+1)^2}{4} \right\rfloor & \text{if } n < 3r, \\ (n-r)(2r+1) & \text{if } n \geq 3r. \end{cases}$$

Theorem 3.1 Let $f(n, r)$ and $f'(n, r)$ be defined as above, where $n > r+1$, then $f(n, r) \geq f'(n, r)$.

Proof Let $p = \min\{r, \lfloor (n-r)/2 \rfloor\}$. Then $n \geq r+2p$ and $p \leq r$. By Lemma 3.2, the matrix A_p is r -nde. Thus $f(n, r) \geq \sigma(A_p) = (n-p)(r+p+1) = f'(n, r)$. \square

Now we want to show that, for any integer i with $g(n, r) \leq i \leq f'(n, r)$, there is always an r -nde matrix A with order n and $\sigma(A) = i$. We make use of a particular class of matrices from Definition 3.1. Let N be the adjacency

matrix of $\text{Cay}(Z_{n-t}, \{1, 2, \dots, r-t+1\})$. Then Example 3.1 shows that N is $(r-t)$ -nde with $\sigma(N) = (n-t)(r-t+1)$. Let

$$C_t = \left[\begin{array}{c|c} O_{t \times t} & J_{t \times (n-t)} \\ \hline J_{(n-t) \times t} & N \end{array} \right].$$

By Lemma 3.2, C_t is r -nde if $n \geq r+2t$ and $1 \leq t \leq r$. For a fixed integer l with $t+1 \leq l \leq n$, we define a matrix $D = (d_{ij}) \in B_n$ as follows:

$$d_{ij} = \begin{cases} 0 & \text{if either } i = l, j = t \text{ or } i = t, j = l, \\ 1 & \text{if } i = j = l, \\ c_{ij} & \text{otherwise.} \end{cases}$$

We say that D is obtained from C_t by the $\langle l, t \rangle$ -interchanges.

Lemma 3.3 Suppose $n \geq r+2t$, $1 \leq t \leq r$ and $n \geq r+t+2$. Let $D \in B_n$ be obtained from C_t by the $\langle i, t \rangle$ -interchanges where $t+1 \leq i \leq n$. Then D is r -nde .

Proof Suppose on the contrary that D is r -partly decomposable. Then D contains an $s \times (n-r+1-s)$ zero submatrix. Since $n-t-1 \geq r+1$, every row and column of D has at least $r+1$ positive entries. Thus $s \geq 2$ and $n-r+1-s \geq 2$. Since C_t is r -inde and it differs from D only in the (i, t) , (t, i) and (i, i) positions, the zero submatrix contains $d_{i,t}$ or $d_{t,i}$. Without loss of generality, we suppose the zero submatrix contains $d_{i,t}$. But since $d_{i,1} = d_{i,2} = \dots = d_{i,t-1} = 1$, $d_{i,i} = d_{i,i+1} = \dots = d_{i,i+(r-t)+1} = 1$, the $s \times (n-r+1-s)$ zero submatrix is contained in $D[1, 2, \dots, n | 1, 2, \dots, t-1, i, i+1, \dots, i+(r-t)+1]$, where addition is taken modulo $n-t$. However, this implies that the submatrix F obtained from C_t by deleting columns $1, 2, \dots, t-1, t, i, i+1, \dots, i+(r-t)+1$ would contain an $(s+1) \times (n-r-s)$ zero submatrix, since all entries in row $i-1$ of F are zero. This shows that C_t contains the zero submatrix F , a contradiction to the r -indecomposability of C_t . Thus D is r -inde. We observe that, for any non-zero (i, j) entry of D , either row i or column j of D has exactly $(r+1)$ 1's. Therefore D is r -nde. \square

We construct a series of matrices from C_t as follows:

1. The matrix $L_{1,t}$ is obtained from C_t by the $\langle n, t \rangle$ -interchanges.
2. For each $i \in \{1, 2, \dots, i_t - 1\}$ with $i_t = \min\{\lfloor \frac{n-t}{2} \rfloor, n-t-r-1\}$, the matrix $L_{2i+1,t}$ is obtained from $L_{2i-1,t}$ by the $\langle n-2i, t \rangle$ -interchanges.

The following lemma can be proved similarly by applying the same proof techniques shown in Lemma 3.3.

Lemma 3.4 For any $i \in \{1, 2, \dots, i_t\}$ with $i_t = \min\{\lfloor \frac{n-t}{2} \rfloor, n-t-r-1\}$, the matrix $L_{2i-1,t}$ is r -nde and $\sigma(C_t) - \sigma(L_{2i-1,t}) = i$. \square

We define $i_t = \min\{\lfloor \frac{n-t}{2} \rfloor, n-t-r-1\}$ and $i'_t = \begin{cases} i_t - 1 & \text{if } i_t = \lfloor \frac{n-t}{2} \rfloor, \\ i_t & \text{otherwise.} \end{cases}$

Lemma 3.5 Suppose $n \geq r + 2t$, $2 \leq t \leq r$ and $1 \leq i \leq t$. Let D be obtained from $L_{2i-1,t}$ by the $\langle n-2i+1, t-1 \rangle$ -interchanges. Then D is r -nde.

Proof Suppose on the contrary that D is r -partly decomposable. Then D contains an $s \times (n-r+1-s)$ zero submatrix. Since $n-t-i_t \geq r+1$, every row and column of D has at least $r+1$ positive entries. Also one can easily verify that every $2 \times n$ submatrix of D has at least $r+2$ nonzero columns and every $n \times 2$ submatrix of D has at least $r+2$ nonzero rows. Thus $s \geq 3$ and $n-r+1-s \geq 3$. Since $L_{2i-1,t}$ is r -inde and it differs from D only in the $(n-2i+1, t-1)$, $(t-1, n-2i+1)$ and $(n-2i+1, n-2i+1)$ positions, the zero submatrix contains $d_{n-2i+1,t-1}$ or $d_{t-1,n-2i+1}$. Without loss of generality, we suppose the zero submatrix contains $d_{n-2i+1,t-1}$. But since $d_{n-2i+1,1} = d_{n-2i+1,2} = \dots = d_{n-2i+1,t-2} = d_{n-2i+1,t} = 1$, $d_{n-2i+1,n-2i+1} = d_{n-2i+1,n-2i+1+1} = \dots = d_{n-2i+1,n-2i+1+(r-t)+1} = 1$, the $s \times (n-r+1-s)$ zero submatrix is contained in $D[1, 2, \dots, n | 1, 2, \dots, t-2, t, n-2i+1, n-2i+1+1, \dots, n-2i+1+(r-t)+1]$. However, this implies that the submatrix F obtained from $L_{2i-1,t}$ by deleting columns $1, 2, \dots, t-2, t-1, t, n-2i, n-2i+1, n-2i+1+1, \dots, n-2i+1+(r-t)+1$ would contain an $(s+2) \times (n-r-s-1)$ zero submatrix, since all entries in rows $n-2i$ and $n-2i-1$ of F are zero. This is a contradiction to the r -indecomposability of $L_{2i-1,t}$. Thus D is r -inde. Furthermore, D is r -nde since, for each non-zero (i, j) entry of D , either row i or column j has exactly $(r+1)$ 1's. \square

We construct a series of r -nde matrices from $L_{2i-1,t}$ as follows:

1. The matrix $L_{2,t}$ is obtained from $L_{2i-1,t}$ by the $\langle n-1, t-1 \rangle$ -interchanges.
2. For each $i \in \{1, 2, \dots, i'_t - 1\}$, the matrix $L_{2i+2,t}$ is obtained from $L_{2i,t}$ by the $\langle n-2i-1, t-1 \rangle$ -interchanges.

Lemma 3.6 Suppose $n \geq r + 2t$, $2 \leq t \leq r$ and $1 \leq i \leq t$. Then $L_{2i,t}$ is

r -nde and

$$\sigma(C_t) - \sigma(L_{2i'_t,t}) = i_t + i'_t = \begin{cases} 2\lfloor \frac{n-t}{2} \rfloor - 1 & \text{if } \lfloor \frac{n-t}{2} \rfloor \leq n-t-r-1, \\ 2(n-t-r-1) & \text{if } \lfloor \frac{n-t}{2} \rfloor > n-t-r-1. \end{cases}$$

□

For $n \geq r + 2t$ with $t = 1$, we have the following construction:

1. The $L_{1,1}$ is obtained from C_1 by the $\langle n, 1 \rangle$ -interchanges.
2. For each $i \in \{1, 2, \dots, n-r-3\}$, the matrix $L_{i+1,1}$ is obtained from $L_{i,1}$ by the $\langle n-i, 1 \rangle$ -interchanges.

Lemma 3.7 Suppose $n \geq r+2t$ with $t = 1$. Then, for each $i \in \{1, 2, \dots, n-r-2\}$, $L_{i,1}$ is r -nde and $\sigma(C_1) - \sigma(L_{i,1}) = i$. □

Brualdi and Hedrick [6, Theorem 3.4] showed that for each i with $g(n, 1) \leq i \leq f'(n, 1)$ and $n \geq 2$, there is always a 1-nde matrix A with order n and $\sigma(A) = i$. Our next theorem extends their result.

Theorem 3.2 For any $r \in \{1, 2, \dots, n-2\}$ and any $i \in [g(n, r), f'(n, r)]$, there exists an r -nde matrix A with order n and $\sigma(A) = i$.

Proof Case 1: $r = 1$. Then, by lemma 3.7, $C_1, L_{1,1}, L_{2,1}, \dots, L_{n-3,1}$ are all 1-nde matrices and the number of positive entries in these matrices cover the entire interval from $2n$ to $3n-3$.

Case 2: $r \geq 2$. Let positive integer $p = \min\{r, \lfloor (n-r)/2 \rfloor\}$. Then $n \geq r + 2p$ and $1 \leq p \leq r$. By Lemmas 3.4 and 3.6, the following matrices are all r -nde:

$$\begin{aligned} & C_p, L_{1,p}, L_{3,p}, \dots, L_{2i_{p-1},p}, L_{2,p}, L_{4,p}, \dots, L_{2i'_p,p}; \\ & C_{p-1}, L_{1,p-1}, L_{3,p-1}, \dots, L_{2i_{p-1}-1,p-1}, L_{2,p-1}, L_{4,p-1}, \dots, L_{2i'_{p-1},p-1}; \\ & \dots\dots; \\ & C_2, L_{1,2}, L_{3,2}, \dots, L_{2i_{2-1},2}, L_{2,2}, L_{4,2}, \dots, L_{2i'_2,2}; \\ & C_1, L_{1,1}, L_{2,1}, \dots, L_{n-r-2,1}. \end{aligned}$$

Since

$$\sigma(C_t) - \sigma(C_{t-1}) = n - 2t - r \leq \begin{cases} 2\lfloor \frac{n-t}{2} \rfloor - 1 & \text{if } \lfloor \frac{n-t}{2} \rfloor \leq n-t-r-1, \\ 2(n-t-r-1) & \text{if } \lfloor \frac{n-t}{2} \rfloor > n-t-r-1, \end{cases}$$

by Lemma 3.6, $\sigma(L_{2i'_t,t}) \leq \sigma(C_{t-1})$ for any $2 \leq t \leq p$. So the number of positive entries in the above sequence of matrices cover the entire interval $[g(n, r), f'(n, r)]$. □

4 The exponents of r -inde and r -nde matrices

The product of r -inde matrices behaves nicely with respect to the indecomposability. This quickly leads to some upper bounds on the index of convergence of r -inde matrices.

Lemma 4.1 Suppose $A_1, A_2 \in B_n$ are r_1 -inde and r_2 -inde, respectively. Then the product A_1A_2 is r -inde, where $r = \min\{n-1, r_1+r_2\}$.

Proof Let v be a row vector of length n with all entries non-negative. Let $|v|$ be the number of positive entries in v . Then, by Lemma 2.1(iv), it is easy to see that a matrix A is r -inde if and only if $|vA| \geq \min\{n, |v|+r\}$ for all such v with $|v| > 0$. Since A_1, A_2 are r_1 -inde and r_2 -inde, respectively, we have $|vA_1A_2| \geq \min\{n, |vA_1|+r_2\} \geq \min\{n, |v|+r_1+r_2\}$. Therefore A_1A_2 is r -inde, where $r = \min\{n-1, r_1+r_2\}$. \square

Corollary 4.1 Suppose $r \geq 1$ and A is r -inde. Then $\exp(A) \leq \lceil (n-1)/r \rceil$.

Proof Let $k = \lceil (n-1)/r \rceil$. Then $rk \geq n-1$. Since A is r -inde, by Lemma 4.1, the matrix A^k is $\min\{n-1, rk\} = (n-1)$ -inde. Thus $A^k = J$. \square

The following corollary follows from Theorem 2.4 and Corollary 4.1.

Corollary 4.2 Let $A = \{a_1, \dots, a_k\}$ be a subset of an Abelian group G . Suppose $\text{Cay}(G, A)$ is primitive. Then $\exp(\text{Cay}(G, A)) \leq \lceil \frac{n-1}{\lfloor k/2 \rfloor} \rceil$. \square

Let

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}_{n \times n} .$$

A circulant Boolean matrix is a matrix of the form $C = P^{a_1} + P^{a_2} + \dots + P^{a_k}$ ($0 \leq a_1 < a_2 < \dots < a_k < n$). We denote it by $C\langle a_1, a_2, \dots, a_k; n \rangle$ for convenience. The set of all circulants of order n forms a multiplicative semigroup C_n with $|C_n| = 2^n$. The following corollary was originally proved by Huang by using several lemmas. Now we can see that it follows immediately from Corollary 4.2 since the associated digraph of $C\langle a_1, a_2, \dots, a_k; n \rangle$ is $\text{Cay}(Z_n, \{a_1, a_2, \dots, a_k\})$

Corollary 4.3 (Huang [11, Theorem 2.1]) Suppose $C = C\langle a_1, a_2, \dots, a_k; n \rangle$ is primitive. Then either $\exp(C) = n-1$ or $\exp(C) \leq \lfloor \frac{n}{2} \rfloor$. \square

5 Further research

We proved $f(n, r) \geq f'(n, r)$ in Theorem 3.1. For $r = 1$ and $n \geq 2$, Brualdi and Hedrick [6, Theorems 3.3, 3.4] confirmed that $f(n, r) = f'(n, r)$. Now it is natural to ask the following question:

Question Does $f(n, r) = f'(n, r)$ hold for all $n > r + 1, r \geq 2$?

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