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Bounds on the local bases of primitive nonpowerful nearly reducible sign patterns

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In this work, we study the kth local base, which is a generalization of the base, of a primitive non-powerful nearly reducible sign pattern of order $n \geq 7$. We obtain the sharp bound together with a complete characterization of the equality case, of the kth local bases for primitive non-powerful nearly reducible sign patterns. We also show that there exist "gaps" in the kth local base set of primitive non-powerful nearly reducible sign patterns.

Keywords: sign pattern; nearly reducible; primitive; powerful; local base

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1. Introduction

A *sign pattern (matrix) A* is a matrix whose entries are from the set $\{1, -1, 0\}$. In the computations of the powers of a square sign pattern, an ambiguous sign may arise when a positive sign is added to a negative sign. So a new symbol # was introduced in [4] to denote such an ambiguous sign. Following [4], we call a matrix with entries from the set $\Gamma = \{0, 1, -1, \#\}$ a *generalized sign pattern (matrix)*. Addition and multiplication involving the symbol # are defined as follows:

$$(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \# \text{ (for all } a \in \Gamma);$$
$$0 \cdot \# = \# \cdot 0 = 0; \quad b \cdot \# = \# \cdot b = \# \text{ (for all } b \in \Gamma\backslash\{0\}).$$

For a sign pattern A, we use $|A|$ to denote the (0, 1)-matrix obtained from A by replacing each nonzero entry by 1. Clearly $|A|$ completely determines the zero pattern of A. A square nonnegative matrix A is *primitive* if some power $A^k > 0$ (that is, A^k is entrywise positive). The least such k is called the *primitive exponent* of A, denoted by $\exp(A)$. A square sign pattern A is called primitive if $|A|$ is primitive, and in this case we define $\exp(A) = \exp(|A|)$. A square sign pattern A is called *powerful* if each power of A contains no # entry.

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Definition 1.1 [6,9] Let A be a sign pattern of order n and A, A^2, $A^3, \ldots$ be the sequence of powers of A. Since there are only 4^{n^2} different generalized sign patterns of order n, there exist repetitions in the above sequence. Let l be the least positive integer satisfying $A^l = A^{l+p}$ for some positive integer p. Then l is called the *generalized base* (or simply *base*) of A, denoted by $l(A)$.

It was shown in [9] that if a square sign pattern A is primitive nonpowerful, then $l(A)$ is the least positive integer such that each entry of $A^{l(A)}$ is #.

Definition 1.2 Let A be a primitive nonpowerful sign pattern of order n and $1 \le k \le n$. The *kth local base* of A, denoted by $l_A(k)$, is defined to be the least positive integer h such that A^h has at least k rows each of whose entries is #.

It is well-known that the graph theoretical methods are often useful in the study of the power of square matrices, so we now introduce some graph theoretical concepts.

Let D be a digraph (permits loops but no multiple arcs). A *walk* W in a digraph is a sequence of arcs, $e_1, e_2, \ldots, e_k$, such that the terminal vertex of e_i is the initial vertex of e_{i+1} for $i = 1, \ldots, k-1$. The number k of arcs is called the *length* of the walk W, denoted by $l(W)$. The digraph D is called *primitive* if there is a positive integer k such that for all ordered pairs of vertices x and y (not necessarily distinct) in D, there exists a walk of length k from x to y. The least such k is called the *primitive exponent* of D, denoted by $\exp(D)$. As we know, a digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of all the cycles of D is 1 [3].

Let D be a primitive digraph. For any $v_i, v_j \in V(D)$, the *exponent from v_i to v_j*, denoted by $\exp_D(v_i, v_j)$, is the smallest positive integer m such that there is a walk of length t from v_i to v_j for each $t \ge m$. The *exponent of vertex $v_i \in V(D)$*, denoted by $\exp_D(v_i)$, is the smallest positive integer m such that there is a walk of length t from v_i to each vertex of D for each $t \ge m$. Clearly, $\exp_D(v_i) = \max\{\exp_D(v_i, v_j) | v_j \in V(D)\}$. If we choose to order the vertices of D in such a way that

$$\exp_D(1) \le \exp_D(2) \le \cdots \le \exp_D(n),$$

then $\exp_D(k)$ is called the *kth local exponent* of D (i.e., the kth smallest vertex exponent). It is clear that $\exp(D) = \exp_D(n)$.

A *signed digraph* S is a digraph where each arc of S is assigned a sign 1 or -1. The *sign* of the walk W (in a signed digraph), denoted by sgn W, is defined to be the product of signs of all arcs in W. Two walks W_1 and W_2 in a signed digraph are called *a pair of SSSD walks*, if they have the same initial vertex, same terminal vertex, same length, but different signs. A signed digraph S is called *powerful* if S contains no pairs of *SSSD* walks, and is called *primitive* if its nonsigned digraph is primitive.

Definition 1.3 Let S be a primitive nonpowerful signed digraph of order n. The *base* of S, denoted by $l(S)$, is defined to be the smallest positive integer h such that for any vertex v_i and vertex v_j (not necessarily distinct) in S, there exists a pair of *SSSD* walks of length h from v_i to v_j. For any $v_i, v_j \in V(S)$, the *base from v_i to v_j*, denoted by $l_S(v_i, v_j)$, is defined to be the smallest positive integer l such that there is a pair of *SSSD* walks of length t from v_i to v_j for each integer $t \ge l$. The *base of a vertex $v_i \in V(S)$*, denoted by $l_S(v_i)$, is defined to be the smallest positive integer l such that there is a pair of *SSSD* walks of length t from v_i to each vertex $v_i \in V(S)$ for each integer $t \ge l$. We choose to order the vertices of S in such a way that

$$l_S(1) \le l_S(2) \le \cdots \le l_S(n),$$

then we call $l_S(k)$ the *kth local base* of S (i.e., the kth smallest vertex base).

Clearly, $l_S(v_i) = \max\{l_S(v_i, v_j)|v_j \in V(S)\}$, and $l(S) = \max\{l_S(v_i)|v_i \in V(S)\}$.

Let $A=(a_{ij})$ be a sign pattern of order n. The *associated digraph* $D(A)$ of A is defined to be the digraph with vertex set $V=\{v_1, v_2, \dots, v_n\}$ and arc set $E = \{(v_i, v_j)|a_{ij} \neq 0\}$. The *associated signed digraph* $S(A)$ of A is obtained from $D(A)$ by assigning the sign of each a_{ij} to the arc (v_i, v_j) in $D(A)$. Let S be a signed digraph of order n. Then there is a sign pattern A of order n whose associated signed digraph $S(A)$ is S. It is easy to see from the above relation between sign patterns and sign digraphs that a sign pattern A of order n is powerful if and only if its associated signed digraph $S(A)$ is powerful, A is primitive if and only if $S(A)$ is primitive, $l(A)=l(S(A))$, and $l_A(k)=l_{S(A)}(k)$ for $k=1,2,\dots,n$.

A square matrix A is *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & 0 \\ D & C \end{bmatrix},$$

where B and C are square nonvacuous matrices. The matrix A is *irreducible* if it is not reducible, and is *nearly reducible* (or simply NR) if it is irreducible and each matrix obtained from A by replacing a nonzero entry by 0 is reducible. A sign pattern A is called *nearly reducible* if $|A|$ is nearly reducible. Clearly for a sign pattern A, $|A|$ is NR if and only if $D(A)$ is a minimally strongly connected digraph (or simply a NR digraph), and A is NR if and only if $S(A)$ is a minimally strongly connected signed digraph (or simply a NR signed digraph).

In this work, we shall study the kth local base of a primitive nonpowerful NR sign pattern (equally, the kth local base of a primitive nonpowerful NR signed digraph) of order $n \geq 7$. We obtain the sharp bound together with a complete characterization of the equality cases, of the kth local bases for primitive nonpowerful NR sign patterns of order n. We also show that there exist "gaps" in the kth local base set of primitive nonpowerful NR sign patterns of order n.

2. Some preliminaries

The following Lemmas 2.1–2.5 are only concerned with the nonsigned digraphs, but these results will be used in our study of the signed digraphs.

LEMMA 2.1 [8] *Let D be a primitive digraph of order n with the set of cycle lengths L. Suppose $\{p, q\} \subseteq L$, where gcd $(p, q)=1$ and $p<q$. If D contains a p-cycle that intersects a q-cycle, then for all $1 \leq k \leq n$, $exp_D(k) \leq p(q-2)+n-q+k$.*

LEMMA 2.2 [1] *Let D be a primitive digraph of order n. Then*

$$\exp_D(k) \leq \exp_D(k-1)+1 \quad \textit{for} \quad 2 \leq k \leq n.$$

Let $r_1<r_2<\cdots<r_t$ be positive integers. The Frobenius–Schur index $\phi(r_1, r_2, \dots, r_t)$ is the least integer such that the equation $x_1r_1+x_2r_2+\cdots+x_rt_r=m$ has a solution in nonnegative integers $x_1, x_2, \dots, x_t$ for all m $\geq \phi(r_1, r_2, \dots, r_t)$. A well-known result due to Schur shows that $\phi(r_1, r_2, \dots, r_t)$ is well defined provided gcd$(r_1, r_2, \dots, r_t)=1$. For $t=2$, it is also well-known that ϕ $(r_1, r_2)=(r_1-1)$ (r_2-1).

In the remainder of this article, let $D_{n-1,s}$ be the primitive NR digraph of order n as given in Figure 1.

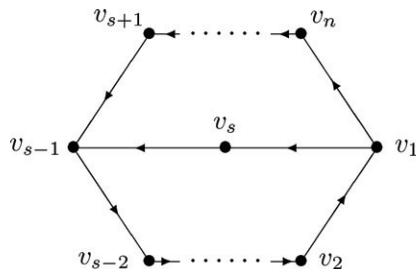

Figure 1. Digraph $D_{n-1,s}$ ($n \geq 7$, $s \leq n-1$).

LEMMA 2.3 [2,7] *Let D be a primitive NR digraph of order $n \geq 7$. Then*

(1) $\exp(D) \leq n^2 - 4n + 6$, *with equality if and only if D is isomorphic to* $D_{n-1,n-2}$.
(2) *For each integer k with* $n^2 - 5n + 9 < k < n^2 - 4n + 6$ *or* $n^2 - 6n + 12 < k < n^2 - 5n + 9$, *there is no primitive NR digraph D of order n with* $\exp(D) = k$.
(3) *Up to isomorphism, there exists zero or one* ($D_{n-1,n-3}$) *primitive NR digraph of order n such that* $\exp(D) = n^2 - 5n + 9$, *according to whether n is odd or even.*

LEMMA 2.4 [5] *For* $n \geq 7$,

$$\exp_{D_{n-1,n-2}}(k) = \exp_{D_{n-1,n-2}}(v_k) = \begin{cases} n^2 - 5n + 7 + k & \text{if } 1 \leq k \leq n-2, \\ n^2 - 4n + 5 & \text{if } k = n-1, \\ n^2 - 4n + 6 & \text{if } k = n. \end{cases}$$

LEMMA 2.5 *For $n \geq 7$ and n is even,*

$$\exp_{D_{n-1,n-3}}(k) = \exp_{D_{n-1,n-3}}(v_k) = \begin{cases} n^2 - 6n + 10 + k & \text{if } 1 \leq k \leq n-3, \\ n^2 - 5n + 7 & \text{if } k = n-2, \\ n^2 - 5n + 8 & \text{if } k = n-1, \\ n^2 - 5n + 9 & \text{if } k = n. \end{cases}$$

Proof For $2 \leq k \leq n-3$ and $1 \leq i \leq n$, it is easy to see that each walk from v_k to v_i of length $l \geq 1$ must contain the vertex v_{k-1}, and so if there exists a walk from v_k to v_i of length $l \geq 1$, then there exists a walk from v_{k-1} to v_i of length $l-1$. Thus,

$$\exp_{D_{n-1,n-3}}(v_{k-1}) \leq \exp_{D_{n-1,n-3}}(v_k) \quad \text{for} \quad 2 \leq k \leq n-3.$$

Similarly, we have

$$\exp_{D_{n-1,n-3}}(v_{n-4}) \leq \exp_{D_{n-1,n-3}}(v_{n-2}) \leq \exp_{D_{n-1,n-3}}(v_{n-1}) \leq \exp_{D_{n-1,n-3}}(v_n). \tag{2.1}$$

It is clear that

$$\exp_{D_{n-1,n-3}}(v_{n-3}) = \exp_{D_{n-1,n-3}}(v_{n-2}). \tag{2.2}$$

Therefore

$$\exp_{D_{n-1,n-3}}(k) = \exp_{D_{n-1,n-3}}(v_k) \quad \text{for} \quad 1 \leq k \leq n. \tag{2.3}$$

By Lemma 2.3

$$\exp_{D_{n-1,n-3}}(n) = \exp_{D_{n-1,n-3}}(v_n) = \exp(D_{n-1,n-3}) = n^2 - 5n + 9. \quad (2.4)$$

Let $1 \le k \le n-3$. By Lemma 2.1, $\exp_{D_{n-1,n-3}}(v_k) \le (n-3)^2 + 1 + k = n^2 - 6n + 10 + k$. On the other hand, let W be any walk of length $n^2 - 6n + 9 + k$ from v_k to v_{n-2}. Then, W is a union of the unique path from v_k to v_{n-2} of length $k+2$ and several cycles of length $n-1$ and several cycles of length $n-3$. Thus, there exist nonnegative integers a, b such that

$$n^2 - 6n + 9 + k = a(n-1) + b(n-3) + k + 2,$$

that is,

$$(n-2)(n-4) - 1 = n^2 - 6n + 7 = a(n-1) + b(n-3).$$

It is contradicting to the definition of $\phi(n-1, n-3)$. Thus, there is no walk of length $n^2 - 6n + 9 + k$ from v_k to v_{n-2}, and so $\exp_{D_{n-1,n-3}}(v_k) \ge n^2 - 6n + 10 + k$. Then

$$\exp_{D_{n-1,n-3}}(k) = \exp_{D_{n-1,n-3}}(v_k) = n^2 - 6n + 10 + k \quad \text{for} \quad 1 \le k \le n-3. \quad (2.5)$$

Combining (2.1)–(2.5), and Lemma 2.2, the result now follows. ■

LEMMA 2.6 [6] *Let S be a primitive signed digraph. Then, S is nonpowerful if and only if S contains a pair of cycles C_1 and C_2 (say, with lengths p_1 and p_2, respectively) satisfying one of the following two conditions*:

(1) *p_1 is odd, p_2 is even, and $sgn\, C_2 = -1$;*
(2) *Both p_1 and p_2 are odd and $sgn\, C_1 = -sgn\, C_2$.*

For convenience, we call a pair of cycles C_1 and C_2 satisfying (1) or (2) in Lemma 2.6 a *distinguished cycle pair*. If C_1 and C_2 form a distinguished cycle pair of lengths p_1 and p_2, respectively, then the closed walks $W_1 = p_2 C_1$ (walk around C_1, p_2 times) and $W_2 = p_1 C_2$ have the same length $p_1 p_2$, but with different signs since $(\text{sgn} C_1)^{p_2} = -(\text{sgn} C_2)^{p_1}$.

LEMMA 2.7 *Let S be a primitive nonpowerful signed digraph with D as its underlying digraph, and $u \in V(S)$. If there is a pair of SSSD walks with length r from vertex u to u, then*

$$l_S(u) \le \exp_D(u) + r.$$

Proof Let v be any vertex of S, W_1 and W_2 be a pair of *SSSD* walks with length r from u to u. For each integer $l \ge \exp_D(u) + r$, $l - r \ge \exp_D(u)$. So, there exists a walk W from u to v of length $l - r$. Then $W_1 + W$ and $W_2 + W$ form a pair of *SSSD* walks of length l from u to v. By the definition of $l_S(u)$, the result follows. ■

LEMMA 2.8 *Let S be a primitive nonpowerful signed digraph of order n. Then*

$$l_S(k) \le l_S(k-1) + 1 \quad \text{for} \quad 2 \le k \le n.$$

Proof Without loss of generality, let $V_1 = \{v_1, v_2, \ldots, v_{k-1}\} \subseteq V(S)$, such that vertex v_i has the ith local base, $1 \le i \le k-1$. Because S is strongly connected, there is a vertex

$x \in V \backslash V_1$ and $y \in V_1$ such that (x, y) is an arc of S. It follows that $l_S(x) \le l_S(y) + 1 \le l_S(k-1) + 1$, and the lemma holds. ■

Lemma 2.9 [6] *Let S be a primitive, nonpowerful, NR signed digraph of order* $n \ge 7$. *Then*

(1) $l(S) \le 2n^2 - 7n + 8$, *with equality if and only if the underlying digraph is isomorphic to* $D_{n-1,n-2}$.
(2) *For each integer* k *with* $2n^2 - 9n + 12 < k < 2n^2 - 7n + 8$ *or* $2n^2 - 11n + 18 < k < 2n^2 - 9n + 12$, *there is no primitive nonpowerful NR signed digraph S of order n with* $l(S) = k$.
(3) $l(S) = 2n^2 - 9n + 12$ *if and only if n is even and the underlying digraph of S is isomorphic to* $D_{n-1,n-3}$; *and there is no primitive, nonpowerful, NR signed digraph S of order n with* $l(S) = 2n^2 - 9n + 12$ *if n is odd.*

3. Main results

Theorem 3.1 *Let S be a primitive, nonpowerful, NR signed digraph of order* $n \ge 7$ *with* $D_{n-1,n-2}$ *as its underlying digraph. Then*

$$l_S(k) = l_S(v_k) = \begin{cases} 2n^2 - 8n + 9 + k & \text{if } 1 \le k \le n-2, \\ 2n^2 - 7n + 7 & \text{if } k = n-1, \\ 2n^2 - 7n + 8 & \text{if } k = n. \end{cases}$$

Proof Firstly, we show that $l_S(v_i) = 2n^2 - 8n + 9 + i$ for $1 \le i \le n-3$.

Because S is primitive nonpowerful and S has only two cycles of lengths $n-1$ and $n-2$ respectively (we denote them C_{n-1} and C_{n-2} respectively), by Lemma 2.6, C_{n-1} and C_{n-2} form a distinguished cycle pair and $\mathrm{sgn}((n-1)C_{n-2}) = -\mathrm{sgn}((n-2)C_{n-1})$. Thus, there is a pair of *SSSD* walks of length $(n-1)(n-2)$ from v_i to v_i for $1 \le i \le n-3$. Then by Lemmas 2.4 and 2.7, $l_S(v_i) \le \exp_{D_{n-1,n-2}}(v_i) + (n-1)(n-2) = 2n^2 - 8n + 9 + i$ for $1 \le i \le n-3$.

On the other hand, we will show that there is no pair of *SSSD* walks of length $2n^2 - 8n + 8 + i$ from v_i to v_{n-1} in S.

Let W_1 and W_2 be any two walks of length $2n^2 - 8n + 8 + i$ from v_i to v_{n-1} in S. Then each W_j $(j = 1, 2)$ is a union of the unique path from v_i to v_{n-1} of length $i+1$ and several cycles of length $n-1$ and several cycles of length $n-2$. Thus, there exist nonnegative integers a_j, b_j $(j = 1, 2)$ such that

$$2n^2 - 8n + 8 + i = a_j(n-1) + b_j(n-2) + i + 1, \quad (j = 1, 2),$$

that is,

$$2n^2 - 8n + 7 = a_j(n-1) + b_j(n-2), \quad (j = 1, 2).$$

So $(a_2 - a_1)(n-1) = (b_1 - b_2)(n-2)$. Let $b_1 - b_2 = (n-1)x$. Then $a_2 - a_1 = (n-2)x$. We claim that $x = 0$.

If $x \ge 1$, then $a_2 \ge n-2$, and so $\phi(n-1, n-2) - 1 = (n-2)(n-3) - 1 = n^2 - 5n + 5 = (a_2 - (n-2))(n-1) + b_2(n+2)$. It is contradicting to the definition of $\phi(n-1, n-2)$. A similar contradiction can be obtained if $x \le -1$. Thus we have $x = 0$. So $a_1 = a_2$, $b_1 = b_2$, and thus $\mathrm{sgn}\, W_1 = \mathrm{sgn}\, W_2$. This shows that there is no pair of *SSSD* walks of length $2n^2 - 8n + 8 + i$ from v_i to v_{n-1} in S. Then $l_S(v_i) \ge 2n^2 - 8n + 9 + i$.

Combining the above two inequalities, we have $l_S(v_i) = 2n^2 - 8n + 9 + i$ for $1 \leq i \leq n-3$.

Second, we show that $l_S(v_{n-2}) = 2n^2 - 7n + 7$.

Since there is an arc from v_{n-2} to v_{n-3}, it is easy to see that $l_S(v_{n-2}) \leq l_S(v_{n-3}) + 1 = 2n^2 - 7n + 7$.

On the other hand, let W_1 and W_2 be any two walks of length $2n^2 - 7n + 6$ from v_{n-2} to v_{n-1} in S. Then, each W_j ($j = 1, 2$) is a union of the unique path from v_{n-2} to v_{n-1} of length $n-1$ and some cycles. Thus, there exist nonnegative integers a_j, b_j ($j = 1, 2$) such that

$$2n^2 - 7n + 6 = a_j(n-1) + b_j(n-2) + (n-1), \quad (j = 1, 2),$$

that is,

$$2n^2 - 8n + 7 = a_j(n-1) + b_j(n-2), \quad (j = 1, 2).$$

Similarly to the above discussion, we have that $\operatorname{sgn} W_1 = \operatorname{sgn} W_2$. Then there is no pair of *SSSD* walks of length $2n^2 - 7n + 6$ from v_{n-2} to v_{n-1} in S, and so $l_S(v_{n-2}) \geq 2n^2 - 7n + 7$.

Combining the above two inequalities, $l_S(v_{n-2}) = 2n^2 - 7n + 7$.

Third, we show that $l_S(v_{n-1}) = 2n^2 - 7n + 7$.

Since there is an arc from v_{n-1} to v_{n-3}, it is easy to see that $l_S(v_{n-1}) \leq l_S(v_{n-3}) + 1 = 2n^2 - 7n + 7$.

On the other hand, let W_1 and W_2 be any two walks of length $2n^2 - 7n + 6$ from v_{n-1} to v_{n-1} in S. Then each W_j ($j = 1, 2$) is a union of several cycles of length $n-1$ and several cycles of length $n-2$. Thus, there exist nonnegative integers a_j, b_j ($j = 1, 2$) such that

$$2n^2 - 7n + 6 = a_j(n-1) + b_j(n-2), \quad (j = 1, 2).$$

Noticing that the vertex v_{n-1} is not in the cycle C_{n-2}, $a_j \geq 1$ for $j = 1, 2$. So $(a_2 - a_1)(n-1) = (b_1 - b_2)(n-2)$. Let $b_1 - b_2 = (n-1)x$. Then $a_2 - a_1 = (n-2)x$. We claim that $x = 0$.

If $x \geq 1$, then $a_2 \geq n-1$ (since $a_1 \geq 1$), and so $\phi(n-1, n-2) - 1 = (n-2)(n-3) - 1 = n^2 - 5n + 5 = (a_2 - (n-1))(n-1) + b_2(n-2)$. It is contradicting to the definition of $\phi(n-1, n-2)$. A similar contradiction can be obtained if $x \leq -1$. Thus, we have $x = 0$. So $a_1 = a_2$, $b_1 = b_2$, and thus $\operatorname{sgn} W_1 = \operatorname{sgn} W_2$. This shows that there is no pair of *SSSD* walks of length $2n^2 - 7n + 6$ from v_{n-1} to v_{n-1} in S. Then $l_S(v_{n-1}) \geq 2n^2 - 7n + 7$.

Combining the above two inequalities, $l_S(v_{n-1}) = 2n^2 - 7n + 7$.

Finally, by Lemma 2.9, we have $l_S(v_n) = 2n^2 - 7n + 8$. Therefore the theorem follows. ■

THEOREM 3.2 *Let n be even and S be a primitive, nonpowerful, NR signed digraph of order $n \geq 7$ with $D_{n-1,n-3}$ as its underlying digraph. Then*

$$l_S(k) = l_S(v_k) = \begin{cases} 2n^2 - 10n + 13 + k & \text{if } 1 \leq k \leq n-3, \\ 2n^2 - 9n + 10 & \text{if } k = n-2, \\ 2n^2 - 9n + 11 & \text{if } k = n-1, \\ 2n^2 - 9n + 12 & \text{if } k = n. \end{cases}$$

Proof First, we show that $l_S(v_i) = 2n^2 - 10n + 13 + i$ for $1 \leq i \leq n-4$.

Because S is primitive nonpowerful and S has only two cycles of lengths $n-1$ and $n-3$ respectively (we denote them C_{n-1} and C_{n-3} respectively), by Lemma 2.6, C_{n-1} and C_{n-3} form a distinguished cycle pair and $\mathrm{sgn}((n-1)C_{n-3}) = -\mathrm{sgn}((n-3)C_{n-1})$. Thus, there is a pair of $SSSD$ walks of length $(n-1)(n-3)$ from v_i to v_i for $1 \le i \le n-4$. Then by Lemmas 2.5 and 2.7, $l_S(v_i) \le \exp_{D_{n-1,n-3}}(v_i) + (n-1)(n-3) = 2n^2 - 10n + 13 + i$, $1 \le i \le n-4$.

On the other hand, we will show that there is no pair of $SSSD$ walks of length $2n^2 - 10n + 12 + i$ from v_i to v_{n-2} in S.

Let W_1 and W_2 be any two walks of length $2n^2 - 10n + 12 + i$ from v_i to v_{n-2} in S. Then each W_j $(j = 1, 2)$ is a union of the unique path from v_i to v_{n-2} of length $i + 2$ and several cycles of length $n-1$ and several cycles of length $n-3$. Thus, there exist nonnegative integers a_j, b_j $(j = 1, 2)$ such that

$$2n^2 - 10n + 12 + i = a_j(n-1) + b_j(n-3) + i + 2, \quad (j = 1, 2),$$

that is,

$$2n^2 - 10n + 10 = a_j(n-1) + b_j(n-3), \quad (j = 1, 2).$$

So $(a_2 - a_1)(n-1) = (b_1 - b_2)(n-3)$. Let $b_1 - b_2 = (n-1)x$. Then $a_2 - a_1 = (n-3)x$. We claim that $x = 0$.

If $x \ge 1$, then $a_2 \ge n-3$, and so $\phi(n-1, n-3) - 1 = (n-2)(n-4) - 1 = n^2 - 6n + 7 = (a_2 - (n-3))(n-1) + b_2(n-3)$. It is contradicting to the definition of $\phi(n-1, n-3)$. A similar contradiction can be obtained if $x \le -1$. Thus we have $x = 0$. So $a_1 = a_2$, $b_1 = b_2$, and thus $\mathrm{sgn}\, W_1 = \mathrm{sgn}\, W_2$. This shows that there is no pair of $SSSD$ walks of length $2n^2 - 10n + 12 + i$ from v_i to v_{n-2} in S. Then $l_S(v_i) > 2n^2 - 10n + 13 + i$.

Combining the above two inequalities, we have that $l_S(v_i) = 2n^2 - 10n + 13 + i$, $1 \le i \le n-4$.

Second, we show that $l_S(v_{n-3}) = 2n^2 - 9n + 10$.

Since there is an arc from v_{n-3} to v_{n-4}, it is easy to see that $l_S(v_{n-3}) < l_S(v_{n-4}) + 1 = 2n^2 - 9n + 10$.

On the other hand, let W_1 and W_2 be any two walks of length $2n^2 - 9n + 9$ from v_{n-3} to v_{n-2} in S. Then each W_j $(j = 1, 2)$ is a union of the unique path from v_{n-3} to v_{n-2} of length $n-1$ and some cycles. Thus, there exist nonnegative integers a_j, b_j $(j = 1, 2)$ such that

$$2n^2 - 9n + 9 = a_j(n-1) + b_j(n-3) + (n-1) \quad (j = 1, 2),$$

that is,

$$2n^2 - 10n + 10 = a_j(n-1) + b_j(n-3) \quad (j = 1, 2).$$

Similarly to the above discussion, we have that $\mathrm{sgn}\, W_1 = \mathrm{sgn}\, W_2$. Then there is no pair of $SSSD$ walks of length $2n^2 - 9n + 9$ from v_{n-3} to v_{n-2} in S, and so $l_S(v_{n-3}) \ge 2n^2 - 9n + 10$.

Combining the above two inequalities, $l_S(v_{n-3}) = 2n^2 - 9n + 10$.

Third, we show that $l_S(v_i) = 2n^2 - 10n + 12 + i$ for $n-2 \le i \le n-1$.

Since there are arcs from v_{n-2} to v_{n-4} and from v_{n-1} to v_{n-2}, respectively, it is easy to see that $l_S(v_{n-2}) \le l_S(v_{n-4}) + 1 = 2n^2 - 9n + 10$, and $l_S(v_{n-1}) \le l_S(v_{n-2}) + 1 = 2n^2 - 9n + 11$.

On the other hand, let W_1 and W_2 be any two walks of length $2n^2 - 10n + 11 + i$ from v_i to v_{n-2} in S. Then each W_j $(j = 1, 2)$ is a union of the unique path from v_i to v_{n-2} of length

$i-n+2$ and several cycles of length $n-1$ and several cycles of length $n-3$. Thus there exist nonnegative integers a_j, b_j ($j=1, 2$) such that

$$2n^2 - 10n + 11 + i = a_j(n-1) + b_j(n-3) + i - n + 2, \quad (j = 1, 2).$$

Noticing that the vertices v_{n-2} and v_{n-1} are not in the cycle C_{n-3}, $a_j \geq 1$ for $j=1, 2$. So $(a_2-a_1)(n-1)=(b_1-b_2)(n-3)$. Let $b_1-b_2=(n-1)x$. Then $a_2-a_1=(n-3)x$. We claim that $x=0$.

If $x \geq 1$, then $a_2 \geq n-2$ (since $a_1 \geq 1$), and so $\phi(n-1, n-3) - 1 = (n-2)(n-4) - 1 = n^2 - 6n + 7 = (a_2 - (n-2))(n-1) + b_2(n-3)$. It is contradicting to the definition of $\phi(n-1, n-3)$. A similar contradiction can be obtained if $x \leq -1$. Thus we have $x=0$. So $a_1=a_2$, $b_1=b_2$, and thus $\operatorname{sgn} W_1 = \operatorname{sgn} W_2$. This shows that there is no pair of *SSSD* walks of length $2n^2-10n+11+i$ from v_i to v_{n-2} in S. Then $l_S(v_i) \geq 2n^2-10n+12+i$.

Combining the above two inequalities, $l_S(v_i)=2n^2-10n+12+i$ for $n-2 \leq i \leq n-1$.

Finally, by Lemma 2.9, we have $l_S(v_n)=2n^2-9n+12$. Therefore the theorem follows. ∎

Lemma 3.3 [7] *Let D be a primitive NR digraph. Then the length of the longest cycle of D does not exceed $n-1$.*

Lemma 3.4 [6] *Let D be a primitive NR digraph and C be a cycle of length $n-1$ in D. Then there only exists a unique cycle of length l ($l<n-1$) satisfying $\gcd(n-1, l)=1$ in D.*

Theorem 3.5 *Let S be a primitive, nonpowerful, NR signed digraph of order $n \geq 7$ with D as its underlying digraph, where D is not isomorphic to $D_{n-1,n-2}$ or $D_{n-1,n-3}$. Then for $1 \leq k \leq n$, we have*

$$l_S(k) \leq 2n^2 - 10n + 10 + k.$$

Proof Since S is primitive nonpowerful, by Lemma 2.6, there is a distinguished cycle pair C_1 and C_2 (with lengths, say, p_1 and p_2, respectively), where p_1C_2 and p_2C_1 have different signs. Note that D is not isomorphic to $D_{n-1,n-2}$ or $D_{n-1,n-3}$. By Lemma 2.3, $\exp(D) \leq n^2-6n+12$.

Case 1 C_1 and C_2 have no common vertices.

Assume that $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$ such that the shortest path P from v_1 to v_2 doesn't contain any vertex of $(V(C_1) \cup V(C_2)) \backslash \{v_1, v_2\}$. Let Q be the shortest path from v_2 to v_1. Denote $l(P)=p$ and $l(Q)=q$. It is easy to see that p_2C_1+P+Q and $P+p_1C_2+Q$ form a pair of *SSSD* walks from v_1 to v_1 of length p_1p_2+p+q. Note $p_1+p_2 \leq n$, $p \leq n-p_1-p_2+1$, and $q \leq n-1$. Then $p_1p_2+p+q \leq p_1p_2 + 2n - p_1 - p_2 = (p_1-1)(p_2-1) + 2n - 1 \leq [(1/2)(p_1+p_2-2)]^2 + 2n - 1 \leq [(1/2)(n-2)]^2 + 2n - 1 = (n^2/4) + n$. By Lemmas 2.7 and 2.8, $l_S(1) \leq l_S(v_1) \leq \exp(D) + p_1p_2 + p + q \leq n^2 - 6n + 12 + (n^2/4) + n \leq 2n^2 - 10n + 11$, and $l_S(k) \leq l_S(1)+k-1 \leq 2n^2-10n+10+k$.

Case 2 C_1 and C_2 have some common vertices. Let $v \in V(C_1) \cap V(C_2)$.

Subcase 2.1 $p_1=p_2=p$.

By Lemma 3.3, $p \leq n-1$. Clearly, p is odd and $\operatorname{sgn} C_1 = -\operatorname{sgn} C_2$. Thus C_1 and C_2 form a pair of *SSSD* walks from v to v of length p. By Lemmas 2.7 and 2.8, $l_S(1) \leq l_S(v) \leq \exp(D) + p \leq n^2 - 6n + 12 + n - 1 = n^2 - 5n + 11$, and $l_S(k) \leq l_S(1) + k - 1 \leq n^2 - 5n + 10 + k < 2n^2 - 10n + 10 + k$.

Subcase 2.2 $p_1 \neq p_2$.

It is easy to see that $p_1 p_2 \leq (n-3)(n-2)$, and $p_2 C_1$ and $p_1 C_2$ form a pair of *SSSD* walks from v to v of length $p_1 p_2$. By Lemmas 2.7 and 2.8, $l_S(1) \leq l_S(v) \leq \exp(D) + p_1 p_2 \leq n^2 - 6n + 12 + (n^2 - 5n + 6) = 2n^2 - 11n + 18$, and $l_S(k) \leq l_S(1) + k - 1 \leq 2n^2 - 11n + 17 + k \leq 2n^2 - 10n + 10 + k$.

Combining the above cases, the result follows. ■

By Theorems 3.1, 3.2 and 3.5, we obtain the following theorem.

Theorem 3.6 *Let S be a primitive, nonpowerful, NR signed digraph of order $n \geq 7$. Denote*

$$l_1(k) = \begin{cases} 2n^2 - 8n + 9 + k & \text{if } 1 \leq k \leq n-2, \\ 2n^2 - 7n + 7 & \text{if } k = n-1, \\ 2n^2 - 7n + 8 & \text{if } k = n, \end{cases}$$

and

$$l_2(k) = \begin{cases} 2n^2 - 10n + 13 + k & \text{if } 1 \leq k \leq n-3, \\ 2n^2 - 9n + 10 & \text{if } k = n-2, \\ 2n^2 - 9n + 11 & \text{if } k = n-1, \\ 2n^2 - 9n + 12 & \text{if } k = n. \end{cases}$$

Then for $1 \leq k \leq n$, we have

(1) $l_S(k) \leq l_1(k)$, *with equality if and only if the underlying digraph of S is isomorphic to* $D_{n-1,n-2}$
(2) *For each integer t with $l_2(k) < t < l_1(k)$ or $2n^2 - 10n + 10 + k < t < l_2(k)$, there is no primitive, nonpowerful, NR signed digraph S of order n with $l_S(k) = t$.*
(3) $l_S(k) = l_2(k)$ *if and only if n is even and the underlying digraph of S is isomorphic to* $D_{n-1,n-3}$; *and there is no primitive, nonpowerful, NR signed digraph S of order n with* $l_S(k) = l_2(k)$ *if n is odd.*

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