

## On the Caccetta–Häggkvist Conjecture

Jian Shen

Department of Mathematics, Southwest Texas State University, San Marcos, TX 78666, USA. e-mail: js48@swt.edu

**Abstract.** It was conjectured by Caccetta and Häggkvist in 1978 that every digraph  $G$  with  $n$  vertices and minimum outdegree at least  $r$  contains a directed cycle of length at most  $\lceil n/r \rceil$ . By refining an argument of Chvátal and Szemerédi, we prove that such  $G$  contains a directed cycle of length at most  $n/r + 73$ .

### 1. Introduction

Let  $G = (V, E)$  denote a digraph on  $n$  vertices. Loops are permitted but no multiple arcs. If  $G$  has at least one (directed) cycle, the minimum length of a cycle in  $G$  is called the *girth* of  $G$ , denoted  $g(G)$ . Let  $\deg_G^+(u)$ , or  $\deg^+(u)$  if  $G$  is specified, denote the outdegree of the vertex  $u$  in  $G$ . Let  $\delta^+(G)$  denote the minimum outdegree of  $G$ . Throughout the paper,  $G$  always denotes a digraph on  $n$  vertices with girth  $g$  and  $\delta^+(G) \geq r$ .

In 1978, Caccetta and Häggkvist [3] proposed the following conjecture:

**Conjecture 1.** *Any digraph on  $n$  vertices with minimum outdegree at least  $r$  contains a directed cycle of length at most  $\lceil n/r \rceil$ .*

This conjecture has been proved for  $r = 2$  by Caccetta and Häggkvist [3], for  $r = 3$  by Hamidoune [5] and for  $r = 4, 5$  by Hoáng and Reed [6]. Recently, Shen [9] proved the conjecture when  $n \geq 2r^2 - 3r + 1$ . This implies that, for each given  $r$ , the number of counterexamples to the conjecture, if any, is finite. While the general conjecture is still open, some weaker statements have been obtained. Among them, it is worth mentioning the following two results of Chvátal and Szemerédi.

**Lemma 1. [4]** *Let  $G$  be a digraph of order  $n$  with  $\delta^+(G) \geq r$ . Then  $g \leq 2n/(r + 1)$ .*

**Lemma 2. [4]** *Let  $G$  be a digraph of order  $n$  with  $\delta^+(G) \geq r$ . Then  $g \leq n/r + 2500$ .*

In 1988, Nishimura [7] refined the proof of Chvátal and Szemerédi, reducing the additive constant in Lemma 2 from 2500 to 304.

We first improve the result in Lemma 1 by showing that  $g \leq 3 \lceil (\ln \frac{2+\sqrt{7}}{3}) \frac{n}{r} \rceil$ . Then by refining the argument of Chvátal and Szemerédi further, we prove that every digraph with  $n$  vertices and minimum outdegree at least  $r$  contains a directed cycle of length at most  $n/r + 73$ .

**2. Main Results**

In [8], we investigated a special case of the Caccetta–Häggkvist Conjecture: What is the smallest possible value  $c$  such that the girth of any digraph on  $n$  vertices with  $\delta^+(G) \geq cn$  is at most 3? The following lemma is an improvement of a recent result of Bondy [2].

**Lemma 3. [8]** *Any digraph on  $n$  vertices with minimum outdegree at least  $(3 - \sqrt{7})n$  has girth  $g \leq 3$ .*

We denote by  $\text{dist}(u, v)$  the number of arcs in a shortest directed path from vertex  $u$  to  $v$  in  $G$ . For any non-negative integer  $i$ , let  $N_i(u) := \{w \in V : \text{dist}(u, w) \leq i\}$  and  $N'_i(u) := \{w \in V : \text{dist}(w, u) \leq i\}$ . Similarly for a subset  $V_1$  of  $V$ , let  $N_i(V_1) := \cup_{u \in V_1} N_i(u)$ . Recall that  $N_0(u) = \{u\}$  as  $u$  is at distance 0 from itself.

**Theorem 1.** *Suppose  $G$  is a digraph of order  $n$  with  $\delta^+(G) \geq r$ . Then*

$$g \leq 3 \left\lceil \left( \ln \frac{2 + \sqrt{7}}{3} \right) \frac{n}{r} \right\rceil.$$

*Proof.* Suppose  $G$  is a counterexample with the minimum number of vertices. It may be supposed that  $r < (3 - \sqrt{7})n$ ; otherwise the theorem follows from Lemma 3 immediately.

**Claim 1.**  $|N_i(u)| \geq n(1 - (1 - r/n)^i) + 1$  for all  $u \in V$  and  $1 \leq i \leq g - 1$ . Since  $|N_1(u)| \geq r + 1$ , Claim 1 holds for  $i = 1$ . Now suppose  $i \geq 2$ ,  $|N_{i-1}(u)| \geq n(1 - (1 - r/n)^{i-1}) + 1$  and  $|N_i(u)| < n(1 - (1 - r/n)^i) + 1$ . Since  $i \leq g - 1$ , there are no arcs from  $N_{i-1}(u)$  to  $u$ . Let  $G_1$  be the subdigraph of  $G$  induced by  $N_{i-1}(u) \setminus \{u\}$ . Then  $\delta^+(G_1) \geq r - |N_i(u) \setminus N_{i-1}(u)|$ .

Since  $G_1$  is not a counterexample to Theorem 1,

$$\begin{aligned}
 g \leq g(G_1) &\leq 3 \left[ \left( \ln \frac{2 + \sqrt{7}}{3} \right) \frac{|N_{i-1}(u) \setminus \{u\}|}{r - |N_i(u) \setminus N_{i-1}(u)|} \right] \\
 &\leq 3 \left[ \left( \ln \frac{2 + \sqrt{7}}{3} \right) \frac{|N_{i-1}(u) \setminus \{u\}| - (|N_{i-1}(u) \setminus \{u\}| - n(1 - (1 - r/n)^{i-1}))}{r - |N_i(u) \setminus N_{i-1}(u)| - (|N_{i-1}(u) \setminus \{u\}| - n(1 - (1 - r/n)^{i-1}))} \right] \\
 &\leq 3 \left[ \left( \ln \frac{2 + \sqrt{7}}{3} \right) \frac{n(1 - (1 - r/n)^{i-1})}{r - |N_i(u) \setminus \{u\}| + n(1 - (1 - r/n)^{i-1})} \right] \\
 &\leq 3 \left[ \left( \ln \frac{2 + \sqrt{7}}{3} \right) \frac{n(1 - (1 - r/n)^{i-1})}{r - n(1 - (1 - r/n)^i) + n(1 - (1 - r/n)^{i-1})} \right] \\
 &= 3 \left[ \left( \ln \frac{2 + \sqrt{7}}{3} \right) \frac{n}{r} \right],
 \end{aligned}$$

a contradiction. Thus Claim 1 follows by induction.

Let  $t = \lceil (\ln \frac{2+\sqrt{7}}{3}) \frac{n}{r} \rceil$ . We may suppose  $t \leq g - 1$ ; otherwise  $g \leq t$  and we are done. By claim 1,  $|N_i(u)| \geq n(1 - (1 - r/n)^t) + 1 \geq (3 - \sqrt{7})n + 1$ . Let  $G_2 = (V, E_2)$  be the digraph having the same vertex set as  $G$  such that  $(u, v) \in E_2$  if and only if  $v \in N_i(u) \setminus \{u\}$  in  $G$ . Then  $\delta^+(G_2) \geq (3 - \sqrt{7})n$ . Thus  $g(G_2) \leq 3$  by Lemma 3. By the definition of  $G_2$ , every arc  $(u, v)$  in  $G_2$  can be replaced by a walk of length at most  $t$  from  $u$  to  $v$  in  $G$ . Therefore  $g \leq 3t$ , from which Theorem 1 follows.  $\square$

**Lemma 4.** For any real numbers  $\epsilon$  and  $t$  such that  $0.5 \leq \epsilon \leq 1$  and  $t \geq 226.3$ , there exists an integer  $m = m(\epsilon) \geq 8$  such that

$$m(t + 71.4) - 2mt(2 - \epsilon) + 2t(m - 1)(2 - \epsilon) \cdot \sqrt[m-1]{(2 - \epsilon)/m} > 0$$

and

$$\frac{73(3 - \epsilon)(a + 1)(t + 71.4)}{71.4a(t + 73 - t\epsilon)} - \frac{73(3 - \epsilon)}{t + 73 - t\epsilon} < \frac{t}{a} - 2,$$

where

$$a = m + \frac{m^2(t + 71.4)}{m(t + 71.4) - 2mt(2 - \epsilon) + 2t(m - 1)(2 - \epsilon) \cdot \sqrt[m-1]{(2 - \epsilon)/m}}.$$

*Proof.* We set

$$m = \begin{cases} \lfloor 28 - 20\epsilon \rfloor & \text{if } 0.85 < \epsilon \leq 1, \\ \lfloor 19.5 - 10\epsilon \rfloor & \text{if } 0.5 \leq \epsilon \leq 0.85. \end{cases}$$

Thus  $8 \leq m \leq 14$ . Let  $x = t/(t + 71.4)$  and  $y = \sqrt[m-1]{(2 - \epsilon)/m}$ . Then  $0.76 \leq 226.3/(226.3 + 71.4) \leq x \leq 1$  and

$$\begin{cases} \sqrt[m-1]{(12 + m)/(20m)} \leq y \leq \sqrt[m-1]{(13 + m)/(20m)} & \text{if } 8 \leq m \leq 10, \\ \sqrt[m-1]{(0.5 + m)/(10m)} \leq y \leq \sqrt[m-1]{(1.5 + m)/(10m)} & \text{if } 11 \leq m \leq 14. \end{cases}$$

Let  $f_1(y) = 1 - 2my^{m-1} + 2(m - 1)y^m$ . Then  $f'_1(y) = 2m(m - 1)y^{m-2}(y - 1) < 0$ . Thus

$$\begin{aligned} & m(t + 71.4) - 2mt(2 - \epsilon) + 2t(m - 1)(2 - \epsilon) \cdot \sqrt[m-1]{(2 - \epsilon)/m} \\ &= m(t + 71.4)(1 - 2x(my^{m-1} - (m - 1)y^m)) \\ &\geq m(t + 71.4) \cdot f_1(y) \\ &\geq \begin{cases} m(t + 71.4) \cdot f_1(\sqrt[m-1]{(13 + m)/(20m)}) & \text{if } 8 \leq m \leq 10, \\ m(t + 71.4) \cdot f_1(\sqrt[m-1]{(1.5 + m)/(10m)}) & \text{if } 11 \leq m \leq 14 \end{cases} \\ &> 0. \end{aligned}$$

To prove the second inequality in Lemma 4, it is equivalent to prove  $(3 - \epsilon)(a + 1) + 71.4a(1 - \epsilon)(2t - 73)/(73(t + 71.4)) - 71.4t(t + 73 - t\epsilon)/(73(t + 71.4)) < 0$ .

*Case 1.*  $t \geq 328.5$ . Since  $71.4(2t - 73)/(73(t + 71.4)) < 1.96$  and  $(t + 73 - t\epsilon)/73 \geq 5.5 - 4.5\epsilon = 4.5my^{m-1} - 3.5$ , it suffices to prove  $(3 - \epsilon)(a + 1) + 1.96a(1 - \epsilon) - 71.4t(4.5my^{m-1} - 3.5)/(t + 71.4) < 0$ ; i.e.,

$$f(x, y) = a(2.96my^{m-1} - 0.96) + my^{m-1} + 1 - 71.4x(4.5my^{m-1} - 3.5) < 0,$$

where  $0.82 \leq 328.5/(328.5 + 71.4) \leq x \leq 1$  and

$$a = m + \frac{m}{1 - 2x(my^{m-1} - (m - 1)y^m)}.$$

Recall that  $1 - 2x(my^{m-1} - (m - 1)y^m) \geq f_1(y) > 0$ . Thus  $f(x, y)$  is continuous for all  $x$  and  $y$ . Since  $f''_{xx}(x, y) > 0$ , we have  $\max\{f(x, y)\} = \max\{f(1, y), f(0.82, y)\}$  for every fixed  $y$ . Thus to find  $\max\{f(x, y)\}$ , we can assume  $x$  to be 1 or 0.82 in the rest of the case.

Now  $f'_y(x, y) = m(m - 1)y^{m-2} \cdot h(x, y)$ , where

$$h(x, y) = 2.96a + 1 + \frac{2mx(2.96my^{m-1} - 0.96)(1 - y)}{(1 - 2xmy^{m-1} + 2x(m - 1)y^m)^2} - 321.3x.$$

Since  $m - 1 \geq my$ , it can be checked that  $h'_y(x, y) > 0$ . Thus

$$h(x, y) \leq \begin{cases} h(x, \sqrt[m]{(13+m)/(20m)}) & \text{if } 8 \leq m \leq 10, \\ h(x, \sqrt[m]{(1.5+m)/(10m)}) & \text{if } 11 \leq m \leq 14. \end{cases} < 0$$

Therefore  $f'_y(x, y) < 0$  and  $\max\{f(x, y)\} = \max\{f(x, \sqrt[t]{(12+i)/(20i)}), f(x, \sqrt[j]{(0.5+j)/(10j)}) : x = 1 \text{ or } 0.82, 8 \leq i \leq 10, 11 \leq j \leq 14\} = f(1, \sqrt[7]{1/8}) < 0$ .

Case 2.  $226.3 \leq t < 328.5$ . Since  $71.4(2t - 73)/(73(t + 71.4)) < 1.43$  and  $(t + 73 - t\epsilon)/73 \geq 4.1 - 3.1\epsilon = 3.1my^{m-1} - 2.1$ , it suffices to prove  $(3 - \epsilon)(a + 1) + 1.43a(1 - \epsilon) - 71.4t(3.1my^{m-1} - 2.1)/(t + 71.4) < 0$ ; i.e.,

$$a(2.43my^{m-1} - 0.43) + my^{m-1} + 1 - 71.4x(3.1my^{m-1} - 2.1) < 0,$$

where  $0.76 \leq x \leq 328.5/(328.5 + 71.4) \leq 0.83$ . This can be proved by a similar argument to that in Case 1 above. Therefore Lemma 4 holds.  $\square$

Now we are ready to prove our main theorem. As in [7], our proof in Theorem 2 below follows the basic ideas of the argument in [4]. However unlike the proof in [7], our major improvement results from the following two changes: (i) Theorem 1 is employed to replace Lemma 1. (ii) All key parameters, such as  $m$ ,  $a$  and  $b$ , are by definition dependent on the value of  $\epsilon_1$ ; at the same time, they are kept as balanced as possible to optimize the upper bound for  $g$ . In addition to these, we also make many other changes that result in some minor improvements. To make the paper self-contained, we include a detailed proof.

**Theorem 2.** *Suppose  $G$  is a digraph of order  $n$  with  $\delta^+(G) \geq r$ . Then*

$$g \leq \frac{n}{r} + 73.$$

*Proof.* Suppose  $G$  is a counterexample with the minimum number of vertices. Then  $G$  is strongly connected. For convenience, we write  $t = n/r$ . By Theorem 1, it may be supposed that  $t \geq 226.3$ . We can also suppose  $r \geq 6$  by the result in [6].

**Claim 1.**  $|N_1(X) \setminus X| > r - |X| \cdot r/n$  for any proper subset  $X \subset V$ . Otherwise suppose  $|N_1(X) \setminus X| \leq r - |X| \cdot r/n$ . Let  $G_1$  be the subdigraph of  $G$  induced by  $X$ . Then  $\delta^+(G_1) \geq r - |N_1(X) \setminus X| \geq |X| \cdot r/n$ . By the choice of  $G$ , we know that  $G_1$  is not a counterexample for Theorem 2. Thus  $g \leq g(G_1) \leq |X|/\delta^+(G_1) + 73 \leq n/r + 73$ , a contradiction to the choice of  $G$ .

**Claim 2.** *There exists some vertex  $x$  such that*

$$\begin{cases} |N_{\lfloor (t+73)/2 \rfloor}(x)| \leq (n+1)/2 & \text{if } (t+73)/2 - \lfloor (t+73)/2 \rfloor \leq 0.5, \\ |N_{\lfloor (t+72)/2 \rfloor}(x)| \leq (2n-r+1)/4 & \text{if } (t+73)/2 - \lfloor (t+73)/2 \rfloor > 0.5. \end{cases}$$

Otherwise suppose Claim 2 fails; i.e., for any vertex  $x$ ,

$$\begin{cases} |N_{\lfloor (t+73)/2 \rfloor}(x)| \geq (n+2)/2 & \text{if } (t+73)/2 - \lfloor (t+73)/2 \rfloor \leq 0.5, \\ |N_{\lfloor (t+72)/2 \rfloor}(x)| \geq (2n-r+2)/4 & \text{if } (t+73)/2 - \lfloor (t+73)/2 \rfloor > 0.5. \end{cases}$$

Then an averaging argument shows the existence of a vertex  $x$  such that

$$\begin{cases} |N'_{\lfloor (t+73)/2 \rfloor}(x)| \geq (n+2)/2 & \text{if } (t+73)/2 - \lfloor (t+73)/2 \rfloor \leq 0.5, \\ |N'_{\lfloor (t+72)/2 \rfloor}(x)| \geq (2n-r+2)/4 & \text{if } (t+73)/2 - \lfloor (t+73)/2 \rfloor > 0.5. \end{cases}$$

If  $(t+73)/2 - \lfloor (t+73)/2 \rfloor \leq 0.5$ , then there exists a vertex  $y$  other than  $x$  such that  $y \in N_{\lfloor (t+73)/2 \rfloor}(x) \cap N'_{\lfloor (t+73)/2 \rfloor}(x)$ . Thus a shortest path from  $x$  to  $y$  and a shortest path from  $y$  to  $x$  combine into a closed walk of length at most  $t+73$ , a contradiction to the choice of  $G$ . Now suppose  $(t+73)/2 - \lfloor (t+73)/2 \rfloor > 0.5$ . It may be supposed that  $N_{\lfloor (t+72)/2 \rfloor}(x)$  is a proper subset of  $V$ ; otherwise  $g \leq \lfloor (t+72)/2 \rfloor + 1 \leq t+73$ . By Claim 1,  $|N_{\lfloor (t+74)/2 \rfloor}(x)| = |N_{\lfloor (t+72)/2 \rfloor}(x)| + |N_1(N_{\lfloor (t+72)/2 \rfloor}(x) \setminus N_{\lfloor (t+72)/2 \rfloor}(x))| \geq r + |N_{\lfloor (t+72)/2 \rfloor}(x)| \cdot (n-r)/n > (2n+r+2)/4$ . Thus there exists a vertex  $z$  other than  $x$  such that  $z \in N_{\lfloor (t+74)/2 \rfloor}(x) \cap N'_{\lfloor (t+72)/2 \rfloor}(x)$ , from which Claim 2 follows similarly as above.

We define  $c$  by

$$\begin{cases} \lfloor (t+73)/2 \rfloor(r-c) = (n+1)/2 & \text{if } (t+73)/2 - \lfloor (t+73)/2 \rfloor \leq 0.5, \\ \lfloor (t+72)/2 \rfloor(r-c) = (2n-r+1)/4 & \text{if } (t+73)/2 - \lfloor (t+73)/2 \rfloor > 0.5. \end{cases}$$

Thus

$$73/(t+73) > c/r \geq 71.4/(t+71.4). \tag{1}$$

By Claim 2, there exists a smallest positive integer  $d$  with the following property:

$$|N_d(s)| \leq d(r-c) \quad \text{for some vertex } s. \tag{2}$$

Indeed, we have  $d \leq \lfloor (t+73)/2 \rfloor$  and  $|N_d(s)| \leq (n+1)/2$ .

**Claim 3.** *With  $s$  as in (2), there are at most  $(i+1)(r-c)$  vertices  $y$  with  $d-i \leq \text{dist}(s,y) \leq d$  for all non-negative integers  $i \leq d-1$ . By the minimality of  $d$ , we have  $|N_{d-i-1}(s)| > (d-i-1)(r-c)$ . Thus Claim 3 follows from (2).*

Let  $U := \{w : \text{dist}(s,w) = d\}$  and  $|U| = \epsilon_1(r-c)$ . Then  $\epsilon_1 \leq 1$  by Claim 3. On the other hand, since  $|N_{d-1}(s)| \leq |N_d(s)| - 1 \leq (n-1)/2$  and  $U = N_d(s) \setminus N_{d-1}(s)$ , by Claim 1 we have  $|U| > r - |N_{d-1}(s)| \cdot r/n \geq 0.5r$ ; i.e.,  $\epsilon_1 \geq 0.5$ . Recall that  $t \geq 226.3$ . By Lemma 4, there exists an integer  $m = m(\epsilon_1) \geq 8$  such that

$$m(t+71.4) - 2mt(2-\epsilon_1) + 2t(m-1)(2-\epsilon_1) \cdot \sqrt[m-1]{(2-\epsilon_1)/m} > 0 \tag{3}$$

and

$$\frac{73(3 - \epsilon_1)(a + 1)(t + 71.4)}{71.4a(t + 73 - t\epsilon_1)} - \frac{73(3 - \epsilon_1)}{t + 73 - t\epsilon_1} < \frac{t}{a} - 2, \tag{4}$$

where

$$a = m + \frac{m^2(t + 71.4)}{m(t + 71.4) - 2mt(2 - \epsilon_1) + 2t(m - 1)(2 - \epsilon_1) \cdot \frac{m^{-1}\sqrt{2 - \epsilon_1}}{m}} > 0. \tag{5}$$

Let

$$b = \frac{1}{(2 - \epsilon_1)(1 - \frac{m^{-1}\sqrt{2 - \epsilon_1}}{m})}. \tag{6}$$

Then  $b > 0$  and  $2 - \epsilon_1 - 1/b > 0$ . We now divide the proof into the following two cases:

*Case 1.* For every vertex  $v$  with  $\text{dist}(s, v) = d - 1$ , there is a vertex  $w$  with  $\text{dist}(v, w) < t/a - 1$  and  $\text{dist}(s, w) \leq d - 3$ . In this case, we consider the following two subsets of  $G$ :

$$Q := \{v : \text{dist}(s, v) = d - 1\},$$

$$P := \{u : \text{dist}(s, u) = d - 2 \text{ and there are at least } (r - c)/b \text{ arcs } uv \text{ with } v \in Q\}.$$

Let  $|Q| = \epsilon_2(r - c)$ . By Claim 3,  $\epsilon_2 \leq 2 - \epsilon_1$  and  $|P| \leq (3 - \epsilon_1 - \epsilon_2)(r - c)$ .

**Claim 4.** *There are a subset  $R$  of  $P$  and a subset  $S$  of  $Q$  such that  $|R| \leq (2 - \epsilon_1 - 1/b)(r - c)/m$ ,  $|S| \leq m$  and such that for every  $u \in P \setminus R$  there is an arc  $uv$  with some  $v \in S$ . To justify the claim, we may assume  $|Q| > m$ , for otherwise we could set  $R = \emptyset$  and  $S = Q$ ; we may also assume  $P \neq \emptyset$ , or else we could set  $R = S = \emptyset$ . Thus  $\epsilon_2 \geq 1/b$  by the definition of  $P$ .*

Now we shall inductively construct a sequence  $R_0, R_1, \dots, R_m$  of subsets of  $P$  and a sequence  $S_0, S_1, \dots, S_m$  of subsets of  $Q$  such that  $|R_i| \leq (3 - \epsilon_1 - \epsilon_2)(r - c)(1 - 1/(b\epsilon_2))^i$ ,  $|S_i| = i$  and such that  $R_i$  consists of those vertices  $u$  in  $P$  for which there are no arcs  $uv$  with  $v \in S_i$ . To initialize, we may set  $R_0 = P$  and  $S_0 = \emptyset$ . When  $R_i$  and  $S_i$  have been constructed, the number of arcs  $uv$  with  $u \in R_i$  and  $v \in Q \setminus S_i$  is at least  $|R_i| \cdot (r - c)/b$ . Hence there is a vertex  $v^* \in Q \setminus S_i$  such that, writing

$$R^* = \{u \in R_i : uv^* \in E\},$$

we have  $|R^*| \geq |R_i| \cdot (r - c)/(b \cdot |Q \setminus S_i|) \geq |R_i|/(b\epsilon_2)$ . By setting  $R_{i+1} = R_i \setminus R^*$  and  $S_{i+1} = S_i \cup \{v^*\}$ , we have  $|R_{i+1}| = |R_i| - |R^*| \leq |R_i| \cdot (1 - 1/(b\epsilon_2))$  and  $|S_{i+1}| = |S_i| + 1$ . Thus, by induction,  $|R_m| \leq (3 - \epsilon_1 - \epsilon_2)(r - c)(1 - 1/(b\epsilon_2))^m$  and  $|S_m| = m$ , where  $1/b \leq \epsilon_2 \leq 2 - \epsilon_1$ .

Let  $f(x) = (3 - \epsilon_1 - x)(1 - 1/(bx))^m$ , where  $1/b \leq x \leq 2 - \epsilon_1 \leq 1.5$ . Then

$$\begin{aligned} f'(x) &= (1 - 1/(bx))^{m-1} (m(3 - \epsilon_1 - x) + x - bx^2)/(bx^2) \\ &\geq (1 - 1/(bx))^{m-1} \left( m + x \left( 1 - \frac{1}{1 - \sqrt[m-1]{(2 - \epsilon_1)/m}} \right) \right) / (bx^2) \\ &\geq (1 - 1/(bx))^{m-1} \left( m - (2 - \epsilon_1) \frac{\sqrt[m-1]{(2 - \epsilon_1)/m}}{1 - \sqrt[m-1]{(2 - \epsilon_1)/m}} \right) / (bx^2) \\ &\geq (1 - 1/(bx))^{m-1} \left( m - 1.5 \frac{\sqrt[m-1]{1.5/m}}{1 - \sqrt[m-1]{1.5/m}} \right) / (bx^2). \end{aligned}$$

Recall that  $m \geq 8$ . It is easy to prove that  $m^m > 1.5(m + 1.5)^{m-1}$ , which implies  $f'(x) \geq 0$ . Therefore  $|R_m| \leq (r - c) \cdot f(\epsilon_2) \leq (r - c) \cdot f(2 - \epsilon_1) = (2 - \epsilon_1 - 1/b)(r - c)/m$ . By setting  $R = R_m$  and  $S = S_m$ , Claim 4 follows.

Next we shall construct a digraph  $H$  by adding certain new arcs to the subdigraph of  $G$  induced by  $N_{d-2}(s) \setminus R$ . These new arcs  $uz$  are added for each vertex  $u$  in  $P \setminus R$ ; their other endpoints run through all the vertices  $z$  in  $H$  for which there is an arc  $uv$  with  $v \in S$  and  $\text{dist}(v, z) < t/a$ . Recall that  $|N_d(s)| \leq (n + 1)/2$ . Thus  $H$  contains fewer than  $n/2$  vertices.

**Claim 5.**  $\delta^+(H) \geq r - (r - c)/b - |R|$ . To see this, let  $u$  be any vertex in  $H$ . If  $\text{dist}(s, u) \leq d - 3$ , then  $\deg_H^+(u) \geq r - |R|$ . If  $\text{dist}(s, u) = d - 2$  and  $u \notin P$ , then there are at most  $(r - c)/b$  arcs from  $u$  to  $Q$  by the definition of  $P$ . Thus  $\deg_H^+(u) \geq r - (r - c)/b - |R|$ .

Finally suppose  $u \in P \setminus R$ . Then there is an arc  $uv$  with  $v \in S$  by Claim 4; there is a vertex  $w$  with  $\text{dist}(v, w) < t/a - 1$  and  $\text{dist}(s, w) \leq d - 3$ ; there are at least  $r$  vertices  $z$  with  $wz \in E$ . At most  $|R|$  of these vertices  $z$  are in  $R$ , and the remaining ones are in  $H$ . Thus, by the construction of  $H$ , we have  $\deg_H^+(u) \geq r - |R|$ . Therefore Claim 5 follows.

By the choice of  $G$ , Theorem 2 holds for  $H$ ; i.e.,  $H$  contains a directed cycle  $C$  of length at most  $|H|/\delta^+(H) + 73$ .

**Claim 6.**  $g \leq |H|/\delta^+(H) + tm/a + 73$ . To see this, it may be supposed that some positive number  $l$  of new arcs  $uz$  (those contained in  $H$  but not in  $G$ ) are included in  $C$ .

Since each of these new arcs  $uz$  in  $H$  can be replaced by an appropriate directed path of length at most  $t/a + 1$  in  $G$  and passing through  $S$ , we can convert  $C$  into a closed directed walk  $C^*$  of length at most  $|H|/\delta^+(H) + lt/a + 73$  in  $G$ . Since  $C^*$  passes through  $S$  at least  $l$  times and since  $|S| \leq m$  (Claim 4), some vertex in  $S$  occurs on  $C^*$  at least  $\lceil l/m \rceil$  times. Thus  $C^*$  breaks down into at least  $\lceil l/m \rceil$  directed cycles in  $G$ ; i.e.,

$$g \leq \frac{|H|/\delta^+(H) + tl/a + 73}{\lceil l/m \rceil} \leq |H|/\delta^+(H) + tm/a + 73.$$

By (1),  $r - c \leq tr/(t + 71.4)$ . Recall that  $t = n/r$ . Thus by (3), (5), (6) and Claims 4, 5, 6,

$$\begin{aligned} g &\leq \frac{n/2}{r - (r - c)/b - (2 - \epsilon_1 - 1/b)(r - c)/m} + \frac{tm}{a} + 73 \\ &\leq \frac{n(t + 71.4)/2}{r(t + 71.4) - tr/b - tr(2 - \epsilon_1 - 1/b)/m} + \frac{tm}{a} + 73 \\ &= \frac{tm(t + 71.4)}{2m(t + 71.4) - 2tm/b - 2t(2 - \epsilon_1 - 1/b)} + \frac{tm}{a} + 73 \\ &= t + 73. \end{aligned}$$

*Case 2.* There is a vertex  $v$  with  $\text{dist}(s, v) = d - 1$  such that no vertex  $w$  has  $\text{dist}(v, w) < t/a - 1$  and  $\text{dist}(s, w) \geq d - 3$ . In this case, let

$$T_i := \{y \in N_i(v) : d - 2 \leq \text{dist}(s, y) \leq d - 1\}.$$

By Claim 3,  $|T_i| \leq 3(r - c) - |U| = (3 - \epsilon_1)(r - c)$  for all  $i$ . Thus, for all sufficiently large  $i$ ,

$$|T_i| < \frac{aci(t + 73 - t\epsilon_1)}{73(a + 1)}. \tag{7}$$

Let  $j \geq 0$  be the largest  $i$  such that (7) fails.

**Claim 7.**  $j < t/a - 2$ . To see this, we have

$$\frac{acj(t + 73 - t\epsilon_1)}{73(a + 1)} \leq |T_j| \leq (3 - \epsilon_1)(r - c).$$

By (1) and (4),

$$\begin{aligned} j &\leq \frac{73r(3 - \epsilon_1)(a + 1)}{ac(t + 73 - t\epsilon_1)} - \frac{73(3 - \epsilon_1)}{t + 73 - t\epsilon_1} \\ &\leq \frac{73(3 - \epsilon_1)(a + 1)(t + 71.4)}{71.4a(t + 73 - t\epsilon_1)} - \frac{73(3 - \epsilon_1)}{t + 73 - t\epsilon_1} < \frac{t}{a} - 2. \end{aligned}$$

Let  $F$  be the subdigraph of  $G$  induced by  $T_j$ . By Claim 7 and the assumption of this case, each arc  $yw$  with  $y \in T_j$  has  $\text{dist}(s, w) \geq d - 2$ , and so either  $w \in T_{j+1}$  or  $\text{dist}(s, w) = d$ . By the maximality of  $j$ , we have  $|T_{j+1} \setminus T_j| \leq ac(t + 73 - t\epsilon_1)/(73(a + 1))$ . Thus  $\delta^+(F) \geq r - |U| - |T_{j+1} \setminus T_j| \geq r - \epsilon_1(r - c) - ac(t + 73 - t\epsilon_1)/(73(a + 1)) \geq c(t + 73 - t\epsilon_1)/(73(a + 1))$ , where the last inequality follows from (1). Since  $F$  is not a counterexample to Theorem 2,

$$\begin{aligned}
 g &\leq g(F) \leq |T_j|/\delta^+(F) + 73 \leq |T_{j+1}|/\delta^+(F) + 73 \\
 &\leq ac(t + 73 - t\epsilon_1)(j + 1)/(73(a + 1)\delta^+(F)) + 73 \\
 &\leq a(j + 1) + 73 < t + 73,
 \end{aligned}$$

where the last inequality follows from Claim 7. This completes the proof of Theorem 2.  $\square$

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