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Edge-connectivity and super edge-connectivity of P_2-path graphs

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Abstract

For a graph G, the P_2-path graph, $P_2(G)$, has for vertices the set of all paths of length 2 in G. Two vertices are connected when their union is a path or a cycle of length 3. We present lower bounds on the edge-connectivity, $\lambda(P_2(G))$ of a connected graph G and give conditions for maximum connectivity. A maximally edge-connected graph is super-λ if each minimum edge cut is trivial, and it is optimum super-λ if each minimum nontrivial edge cut consists of all the edges adjacent to one edge. We give conditions on G, for $P_2(G)$ to be super-λ and optimum super-λ.

MSC: 05C40

Keywords: Path graph; Edge cut; Edge-connectivity; Super edge-connectivity

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1. Introduction

Throughout this paper, all the graphs are *simple*, that is, without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every $v \in V(G)$, $N_G(v)$ denotes the *neighbourhood* of v, that is, the set of all vertices adjacent to v. The *degree* of a vertex v is $\deg(v) = |N_G(v)|$. The minimum degree $\delta(G)$ of the graph G is the minimum degree over all vertices of G.

A graph G is called *connected* if every pair of vertices is joined by a path. An *edge cut* in a graph G is a set T of edges of G such that $G - T$ is not connected. If T is a minimal edge cut of a connected graph G, then, $G - T$ necessarily contains exactly

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two components, so it is usual to denote an edge cut T as $(C,\bar{C})$, where C is a proper subset of $V(G)$ and $(C,\bar{C})$ denotes the set of edges between C and its complement $\bar{C}$. The *edge-connectivity*, $\lambda(G)$, of a graph G is the minimum cardinality of an edge cut of G. A graph G is called n-edge connected if $\lambda(G) \geqslant n$. A minimum edge cut $(C,\bar{C})$ is called trivial if $C = \{v\}$ or $\bar{C} = \{v\}$ for some vertex v of $\deg(v) = \delta(G)$. It is well known that $\lambda(G) \leqslant \delta(G)$. Thus, a graph G is said to be *maximally edge-connected* when $\lambda(G) = \delta(G)$.

Superconnectivity is a stronger measure of connectivity, introduced by Boesch and Tindell [5], whose study has deserved some attention in the last years, see for instance, [2,4,8,17,18]. A maximally edge-connected graph is called *super-λ* if every edge cut $(C,\bar{C})$ of cardinality $\delta(G)$ satisfies that either $|C| = 1$ or $|\bar{C}| = 1$. The study of super-λ graphs has a particular significance in the design of reliable networks, mainly due to the fact that attaining edge-superconnectivity implies minimizing the number of minimum edge cuts (see [4,18]). In order to measure the super edge-connectivity we use the following parameter introduced in [8] (see also [2]).

$$\lambda_1(G) = \min\{|(C,\bar{C})|, (C,\bar{C})\text{is a nontrivial edge cut}\}.$$

Notice that if $\lambda_1(G)\delta(G)$, then $\lambda_1(G)=\lambda(G)$. When $\lambda_1(G) > \delta(G)$ (that is to say, when every edge cut of order δ is trivial) the graph must be super-λ. Therefore, by means of this parameter we can say that a graph G is super-λ if and only if $\lambda_1(G) > \delta(G)$. Thus, we define the *edge-superconnectivity* of the graph as the value of $\lambda_1(G)$. Furthermore, $\lambda_1(G) \leqslant \min\{\deg(u)+\deg(v),\ e=uv \in E(G)\} - 2 = M$. Hence, G is said to be *optimum super-λ*, if every minimum nontrivial edge cut is the set of edges incident with some edge of G. In this case, $\lambda_1(G) = M \geqslant 2\delta(G) - 2$ (see [8]).

The purpose of this paper is to study the edge-connectivity and edge-superconnectivity in a special kind of graphs, the so-called P_2-path graphs. Following the notation that Know and Niepel use, given a graph G, the vertex set of the $P_2(G)$-path graph is the set of all paths of length two of G. Two vertices of $P_2(G)$ are joined by an edge, if and only if, the intersection of the corresponding paths form an edge of G, and their union forms either a cycle or a path of length 3. This means that the vertices are adjacent if and only if one can be obtained from the other by "shifting" the corresponding paths in G. Path graphs were investigated by Broersma and Hoede [6] as a natural generalization of line graphs. A characterization of P_2-path graphs is given in [6,12], some important structural properties of path graphs are presented in [14–16,1] and distance properties of path graphs are studied in [3,10]. Knor and Niepel showed a stronger connection of path graphs to line graphs in [9] by proving, in particular, that $P_2(G)$ is a subgraph of $L^2(G) = L(L(G))$. Results on the edge connectivity of line graphs are given by Chartrand and Stewart [7], and later by Zamfirescu [19]. The edge connectivity and super edge-connectivity of line graphs was studied by Jixiang Meng [17]. The vertex-connectivity of path graphs has been studied in [11,13]. In [10], the following theorem is proved.

Theorem A. *Let G be a connected graph. Then $P_2(G)$ is disconnected if and only if G contains two distinct paths A and B of length two, such that the degrees of both endvertices of A are* 1 *in G.*

From this theorem it follows that if G is a connected graph with at most one vertex of degree one, then $P_2(G)$ is also connected. We will show that if G is a connected graph with $\delta(G) \geqslant 2$, then $\lambda(P_2(G)) \geqslant \delta(G) - 1$. Furthermore, if $\lambda(G) \geqslant 2$, then $\lambda(P_2(G)) \geqslant 2\delta(G) - 2$. Since $\delta(P_2(G)) = 2\delta(G) - 2$ for regular graphs (and $\delta(P_2(G)) \geqslant 2\delta(G) - 2$ in general), this result is best possible at least for regular graphs. Regarding the superconnectivity, we will show that if G is a graph with $\lambda(G) \geqslant 3$ and $\lambda(P_2(G)) = 2\delta(G) - 2$, then G is super-λ and $\lambda_1(P_2(G)) \geqslant 3\delta(G) - 3$. Furthermore, if G is a δ-regular graph with $\lambda(G) \geqslant 4$, then $P_2(G)$ is optimum super-λ, whence $\lambda_1(P_2(G)) \geqslant 4\delta(G) - 6$.

2. Results

Let G be a graph and let a, b, c and u, v, w be two paths in G that induce adjacent vertices in $P_2(G)$. Let us call the edge connecting (abc, uvw) in $P_2(G)$ an *ab-edge*, if (a, b) is the edge common to both abc and uvw. For any given $ab \in E(G)$, let E_{ab}^{a} denote the set of vertices of $P_2(G)$ of the type xab, $x \in N_G(a)\backslash\{b\}$. Analogously, let E_{ab}^{b} denote the set of vertices of $P_2(G)$ of the type aby, $y \in N_G(b)\backslash\{a\}$.

Lemma 2.1. *Let G be a connected graph with $\delta(G) \geqslant 2$. Let $A = (C, \bar{C})$ be an edge cut of $P_2(G)$, and let $(a, b) \in E(G)$. If A contains ab-edges, then it contains at least* $\min\{\deg(a) - 1, \deg(b) - 1\}$ *ab-edges.*

Proof. If A contains ab-edges, then $(E_{ab}^{a} \cup E_{ab}^{b}) \cap C \neq \emptyset$ and $(E_{ab}^{a} \cup E_{ab}^{b}) \cap \bar{C} \neq \emptyset$. Let $|E_{ab}^{a} \cap C| = s_a$, $|E_{ab}^{b} \cap C| = s_b$, $|E_{ab}^{a} \cap \bar{C}| = r_a$ and $|E_{ab}^{b} \cap \bar{C}| = r_b$. Then these numbers must satisfy, $s_a + r_a = \deg(a) - 1$, $s_b + r_b = \deg(b) - 1$, $s_a + s_b \geqslant 1$ and $r_a + r_b \geqslant 1$. Furthermore, the number of ab-edges contained in A is $s_a r_b + s_b r_a$, that is,

$$|A| = |(C, \bar{C})| \geqslant s_a r_b + s_b r_a. \tag{1}$$

If $s_a = 0$, then $s_b \geqslant 1$ and $r_a = \deg(a) - 1$. Hence, (1) implies $|A| \geqslant \deg(a) - 1$ and the result follows. Similarly, if either $s_b = 0$, or $r_a = 0$, or $r_b = 0$ then the result is also true. Therefore, we can assume that $s_a, s_b, r_a, r_b \geqslant 1$. In this case $s_a r_b + s_b r_a \geqslant s_a + r_a = \deg(a) - 1$ and $s_a r_b + s_b r_a \geqslant s_b + r_b = \deg(b) - 1$ and the result follows. □

Lemma 2.2. *Let G be a connected graph with $\delta(G) \geqslant 2$. Let $A = (C, \bar{C})$ be an edge cut of $P_2(G)$, and consider the set A' of edges of G defined by, $ab \in A'$ if and only if A contains ab-edges.*

(a) *If $(yab, abx) \in A$ with $(y, a) \notin A'$ and $(b, x) \notin A'$, then y is not connected with x in $G - A'$.*

(b) *If there exist both $uvw \in C$ and $u'v'w' \in \bar{C}$ with $(u, v) \notin A'$ or $(v, w) \notin A'$, and, $(u', v') \notin A'$ or $(v', w') \notin A'$, then A' is an edge cut of G.*

Proof. (a) Let us assume that $yab \in C$ and $abx \in \bar{C}$. Clearly, there are no ya-edges and bx-edges in A, because $(y, a) \notin A'$ and $(b, x) \notin A'$. Hence $E_{ya}^{y} \subset C$ and $E_{bx}^{x} \subset \bar{C}$, or in other words, $tya \in C$, for every $t \in N_G(y)\backslash\{a\}$, and $bxz \in \bar{C}$, for every $z \in N_G(x)\backslash\{b\}$. First notice that $x \neq y$. Indeed if $x = y$, then y, a, b, y is a triangle in G, which induces a triangle in $P_2(G)$, namely bay, ayb, yba, bay. This gives a path joining bay with yba in $G - A$, (because $(a, y) \notin A'$ and $(y, b) \notin A'$) which is impossible.

Let us show now that there is a contradiction if we suppose that there exists in $G-A'$ a path $Q_1 : y=r_0, r_1, \ldots, r_k=x$. First, notice that if $r_1 \neq a$ and $r_{k-1} \neq b$, then Q_1 induces in $P_2(G)$ the path $Q_1^* : bay, ayr_1, \ldots, r_{k-1}xb, xba$, which is contained in $P_2(G) - A$, because $ya \notin A'$, $bx \notin A'$, and for $1 \leqslant i \leqslant k$, $(r_{i-1}, r_i) \notin A'$. But this is impossible because all paths joining $bay \in C$ and $xba \in \bar{C}$ must contain edges of A. Therefore, suppose that $r_1 = a$ or $r_{k-1} = b$. If $r_1 = a$ and $r_{k-1} \neq b$, then $Q_1 : y = r_0, a, r_2, \ldots, r_k = x$ induces in $P_2(G)$ the path $Q_1^* : tya, yar_2, \ldots, r_{k-1}xb, xba$, where $t \in N_G(y)\backslash\{a\}$. In this path Q_1^* there are no edges from A because $(r_{i-1}, r_i) \notin A'$ for $1 \leqslant i \leqslant k$, and $(b, x) \notin A'$. This is also impossible because all paths joining $tya \in C$ and $xba \in \bar{C}$ must contain edges of A. If $r_1 \neq a$ and $r_{k-1} = b$, then $Q_1 : y = r_0, r_1, \ldots, b, \ r_k = x$ induces in $P_2(G)$ the path $Q_1^* : bay, ayr_1, \ldots, r_{k-2}bx, bxz$, where $z \in N_G(x)\backslash\{b\}$. In this path Q_1^* there are no edges from A because $(r_{i-1}, r_i) \notin A'$ for $1 \leqslant i \leqslant k$, and $(a, y) \notin A'$, which is again impossible because $bay \in C$ and $bxz \in \bar{C}$. Proceeding in a similar way, we find impossible also the case when $r_1 \neq a$ and $r_{k-1} = b$, and then, y must be not connected with x in $G - A'$.

(b) Suppose, without loss of generality, that $uvw \in C$ with $(u, v) \notin A'$, and $u'v'w' \in \bar{C}$ with $(v', w') \notin A'$. Then, $E^u_{uv} \subset C$ and $E^{w'}_{v'w'} \subset \bar{C}$. Notice also that we can assume that $u \neq w'$. Indeed, suppose that $u=w'$. If $v=v'$, then $u'v'w'=u'vu \in \bar{C}$, which is adjacent to any vertex $tuv \in E^u_{uv} \subset C$. This implies that A must contain uv-edges, or in other words, $uv \in A'$, a contradiction with our assumptions. If $v \neq v'$, then $v'w'v = v'uv \in E^u_{uv} \subset C$, which is adjacent to $u'v'w' \in \bar{C}$. This implies that A must contain $v'w'$-edges, contradicting again our assumptions. So, $u \neq w'$. Let us assume that A' is not an edge cut of G. Then we can consider in $G - A'$ a path $Q_2 : u = s_0, s_1, \ldots, s_h = w'$. By a similar argument as in the proof of (a), we can obtain a contradiction. The result follows. □

Corollary 2.1. *Let G be a graph with $\delta(G) \geqslant 2$ and $\lambda(G) \geqslant 2$. Let A be an edge cut of $P_2(G)$. Then there exist two different edges (a, b) and (c, d) in G, such that A contains both ab-edges and cd-edges.*

Proof. Suppose that $A=(C, \bar{C})$ contains only ab-edges, i.e., with the notation of Lemma 2.2, $A' = \{(a, b)\}$. Let us assume that $yab \in C$ and $abx \in \bar{C}$. Since $(y, a) \notin A'$ and $(b, x) \notin A'$, by Lemma 2.2 (b), A' is an edge cut of G, contradicting the assumption $\lambda(G) \geqslant 2$. □

As a direct consequence of Lemma 2.1 and Corollary 2.1, we obtain the following theorem.

Theorem 2.1. *Let G be a connected graph with $\delta(G) \geqslant 2$. Then,*

(a) $\lambda(P_2(G)) \geqslant \delta(G) - 1$,
(b) $\lambda(P_2(G)) \geqslant 2\delta(G) - 2$ *if* $\lambda(G) \geqslant 2$.

Consider the graph G formed by joining two triangles by a path of length 3. It is easy to see that $\delta(G)=2$, $\lambda(G)=1$ and $\lambda(P_2(G))=1$. Hence, Theorem 2.1(a) is best possible for regular graphs of degree at least two. Moreover, since for regular graphs $\delta(P_2(G))=2\delta(G)-2$, then Theorem 2.1(b) is also best possible whenever G is 2-edge-connected regular graph.

The j-iterated P_2-graph is defined as $P_2^j(G)=P_2(P_2^{j-1}(G))$, for $j\geqslant 2$, and $P_2^1(G)=P_2(G)$. Then we get the following corollary.

Corollary 2.2. *Let G be a connected graph with $\delta(G)\geqslant 3$. Then $\lambda(P_2^2(G))\geqslant 4\delta(G)-6$.*

Now, going one step further, we will state that, $P_2(G)$ is, actually, super edge-connected with $\lambda_1(P_2(G))\geqslant 3(\delta(G)-1)$ whenever G is a 3-edge-connected graph with $\delta(P_2(G))=2\delta(G)-2$. Moreover, we will show that $P_2(G)$ is optimum super-λ when G is a δ-regular 4-edge-connected graph.

Lemma 2.3. *Let G be a graph with $\lambda(G)\geqslant 3$. Let A be a nontrivial edge cut of $P_2(G)$. Then there exist three different edges (a,b), (c,d) and (e,f) in G, such that A contains ab-edges, cd-edges and ef-edges.*

Proof. Suppose that $A=(C,\bar{C})$ contain only ab-edges and cd-edges, i.e., with the notation of Lemma 2.2, $A'=\{(a,b),(c,d)\}$. Let us assume $yab\in C$ and $abx\in\bar{C}$. Since $\lambda(G)\geqslant 3$, vertices y and x are connected in $G-A'$. Hence, from Lemma 2.2(b), it follows that either $(y,a)=(c,d)$ or $(b,x)=(c,d)$. Furthermore, since A is nontrivial we have that $|C|\geqslant 2$ and $|\bar{C}|\geqslant 2$. Let us show that there exists $uvw\in C$ with $(u,v)\notin A'$ or $(v,w)\notin A'$, and there exists $u'v'w'\in\bar{C}$ with $(u',v')\notin A'$ or $(v',w')\notin A'$.

Suppose $(y,a)=(c,d)$, or in other words, $A'=\{(a,b),(y,a)\}$. Then $abx\in\bar{C}$ and $(b,x)\notin A'$. If every vertex $uvw\in C$ satisfies that $(u,v)\in A'$ and $(v,w)\in A'$, then $C=\{yab\}$, which is impossible because $|C|\geqslant 2$. Therefore there exists a vertex $uvw\in C$ with either $(u,v)\notin A'$ or $(v,w)\notin A'$.

Suppose that $(b,x)=(c,d)$, that is, $A'=\{(a,b),(b,x)\}$. Then $yab\in C$ and $(y,a)\notin A'$. Now, if every vertex $u'v'w'\in\bar{C}$ satisfies that $(u',v')\in A'$ and $(v',w')\in A'$, then $\bar{C}=\{abx\}$, which is impossible because $|\bar{C}|\geqslant 2$. Therefore, there exists a vertex $u'v'w'\in\bar{C}$ with either $(u',v')\notin A'$ or $(v',w')\notin A'$.

Now, from Lemma 2.2(b), it follows that A' is an edge cut in G, that is $2=|A'|\geqslant\lambda(G)$. This is a contradiction because by hypothesis $\lambda(G)\geqslant 3$. As a consequence there exist three different edges (a,b), (c,d) and (e,f) in G, such that A contains ab-edges, cd-edges and ef-edges. □

Theorem 2.2. *Let G be a graph with $\lambda(G)\geqslant 3$, such that $\delta(P_2(G))=2\delta(G)-2$. Then $P_2(G)$ is super-λ and $\lambda_1(P_2(G))\geqslant 3(\delta(G)-1)$.*

Proof. By Theorem 2.1, and taking into account that $\delta(P_2(G))=2\delta(G)-2$, it follows that $\lambda(P_2(G))=2\delta(G)-2$. If $P_2(G)$ is not super-λ, then there exists a minimum nontrivial edge cut A with $|A|=2\delta(G)-2$. On the other hand, by Lemmas 2.3 and

2.1, we have $|A| \geqslant 3\delta(G) - 3$, which is a contradiction. Therefore, $P_2(G)$ must be super-λ and again by Lemmas 2.3 and 2.1, we obtain that the edge-superconnectivity $\lambda_1(P_2(G))$ is greater than or equal to $3(\delta(G) - 1)$. □

Notice that if $\delta(G) = 3$ and $\delta(P_2(G)) = 2\delta(G) - 2$, then $P_2(G)$ is optimum super-λ, because $\lambda_1(P_2(G)) \geqslant 3(\delta(G) - 1) = 4\delta(G) - 6 = 2\delta(P_2(G)) - 2 = 6$. For graphs with $\delta(G) \geqslant 4$, Theorem 2.2 gives us a lower bound of edge-superconnectivity of $P_2(G)$. Now, we are going to state that if G is a δ-regular and 4-edge-connected graph, then $P_2(G)$ is optimum super-λ.

Theorem 2.3. *Let G be a δ-regular graph with $\lambda(G) \geqslant 4$. Then $P_2(G)$ is optimum super-λ and $\lambda_1(P_2(G)) = 4\delta - 6$.*

Proof. By Theorem 2.2, $P_2(G)$ is super-λ and $\lambda_1(P_2(G)) \geqslant 3(\delta - 1)$. Let us assume that $P_2(G)$ is not optimum super-λ, which means that there exists a minimum nontrivial edge cut $A = (C, \bar{C})$ with $3(\delta - 1) \leqslant |A| \leqslant 4\delta - 7$. Hence, by Lemmas 2.1 and 2.3, A contains only three different a_ib_i-edges, $i = 1, 2, 3$. Furthermore, notice that $|C| \geqslant 4$ and $|\bar{C}| \geqslant 4$. Indeed, since A is nontrivial, then $|C| \geqslant 2$ and $|\bar{C}| \geqslant 2$. If $|C| = 2$ or $|\bar{C}| = 2$, then $|A| \geqslant 4\delta - 6$, and if $|C| = 3$ or $|\bar{C}| = 3$, then $|A| \geqslant 6\delta - 10$, which contradicts that $|A| \leqslant 4\delta - 7$. Set $A' = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ and assume that $ya_1b_1 \in C$ and $a_1b_1x \in \bar{C}$. Since $\lambda(G) \geqslant 4$, vertices y and x are connected in $G - A'$. Hence from Lemma 2.2(a), it follows that either $(y, a_1) \in A'$ or $(b_1, x) \in A'$. Let us show that there exist a vertex $uvw \in C$ such that $(u, v) \notin A'$ or $(v, w) \notin A'$, and a vertex $u'v'w' \in \bar{C}$ such that $(u', v') \notin A'$ or $(v', w') \notin A'$. Then we have the following cases.

Case 1: $A' = \{(a_1, b_1), (y, a_1), (b_1, x)\}$. Suppose that every vertex $uvw \in C$ satisfies $(u, v) \in A'$ and $(v, w) \in A'$. Then if $x \neq y$, it follows that $C = \{ya_1b_1\}$, and if $x = y$, then $C \subset \{ya_1b_1, a_1yb_1\}$, contradicting that $|C| \geqslant 4$. Therefore, there exists a vertex $uvw \in C$ such that $(u, v) \notin A'$ or $(v, w) \notin A'$. Analogously, there exists a vertex $u'v'w' \in \bar{C}$ satisfying that $(u', v') \notin A'$ or $(v', w') \notin A'$.

Case 2: $(y, a_1) \in A' - \{(a_1, b_1)\}$ and $(b_1, x) \notin A'$. In this case we have that $a_1b_1x \in \bar{C}$ and $(b_1, x) \notin A'$, and only the existence of a vertex $uvw \in C$ such that $(u, v) \notin A'$ or $(v, w) \notin A'$ remains to be proved. Since A is a nontrivial edge cut, it follows that $(E^{b_1}_{a_1b_1} \cup E^{y}_{ya_1}) \cap C \neq \emptyset$. Then, for a vertex a_1b_1x' we have $(b_1, x') \in A'$, because otherwise we would have that $a_1b_1x' \in C$ and $(b_1, x') \notin A'$, following that A' is an edge cut of G because of Lemma 2.2(b). But this is impossible because $3 = |A'|$ and $\lambda(G) \geqslant 4$. Therefore, $A' = \{(a_1, b_1), (y, a_1), (b_1, x')\}$. Suppose that every vertex $uvw \in C$ satisfies $(u, v) \in A'$ and $(v, w) \in A'$. Then, if $x' \neq y$, we deduce that $C = \{ya_1b_1, a_1b_1x'\}$, and if $x' = y$, then $C \subset \{ya_1b_1, a_1yb_1, a_1b_1y\}$, which is impossible because $|C| \geqslant 4$. Finally, for a vertex zya_1, reasoning as above we get that $(z, y) \in A'$, hence $A' = \{(a_1, b_1), (y, a_1), (z, y)\}$. Suppose again that every vertex $uvw \in C$ satisfies $(u, v) \in A'$ and $(v, w) \in A'$. Now observe that necessarily z, y, a_1 are three different vertices in G, but b_1 could be equal to z. So, if $b_1 \neq z$, then $C = \{ya_1b_1, zya_1\}$, and if $b_1 = z$, then $C \subset \{ya_1b_1, a_1yb_1, a_1b_1y\}$, again a contradiction because $|C| \geqslant 4$. Therefore, there exists a vertex $uvw \in C$ such that $(u, v) \notin A'$ or $(v, w) \notin A'$.

Case 3: $(b_1,x)\in A'-\{(a_1,b_1)\}$ and $(y,a_1)\notin A'$. In this case we have that $ya_1b_1\in C$ and $(y,a_1)\notin A'$. Now, to show that there exists a vertex $u'v'w'\in\bar{C}$ satisfying that $(u',v')\notin A'$ or $(v',w')\notin A'$, we reason exactly in the same way as in case 2, and so we omit it.

In any case, we can apply Lemma 2.2(b), following that A' must be an edge cut of G, that is, $3=|A'|\geqslant\lambda(G)\geqslant 4$, which is a contradiction. Therefore $P_2(G)$ is optimum super-λ, and $\lambda_1(P_2(G))=4\delta-6$. □

Concerning the j-iterated P_2-graph, from Corollary 2.2, it follows that $\lambda(P_2^2(G))\geqslant 6$, for a given connected graph. So we get the following corollary.

Corollary 2.3. *Let G be a δ-regular connected graph with $\delta\geqslant 3$. Then $P_2^3(G)$ is optimum super-λ and $\lambda_1(P_2^3(G))=16\delta-30$.*

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